General Solution to LaPlace's Equation in Spherical Harmonics (Spherical Harmonic Analysis)

- LaPlace's equation is $\nabla^2 U = 0$, and in rectangular (cartesian) coordinates,
  \[ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \]
- In spherical coordinates, where $r$ is distance from the origin of the coordinate system, $\phi$ is the colatitude, and $\theta$ is azimuth or longitude:
  \[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \lambda^2} = 0 \]
- Solutions to LaPlace's equation are called \textbf{harmonics}
- In spherical coordinates, the solutions would be \textbf{spherical harmonics}
- Example: show that $\nabla^2 U = 0$ for point mass ($U = \frac{GM}{r}$)

Solving LaPlace's Equation

- Assume variables are separable: $U = R(r) \Theta(\theta) \Phi(\lambda)$, so
  \[ \begin{align*}
  \nabla^2 U &= \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \lambda^2} = 0 \\
  \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \lambda^2} &= 0 
  \end{align*} \]
- Multiply through by $\frac{R^2 \sin^2 \theta}{r^2}$:
  \[ \begin{align*}
  \sin^2 \theta \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial R}{\partial r} \right) + \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 L}{\partial \lambda^2} &= 0 \\
  \sin^2 \theta \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial R}{\partial r} \right) + \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) &= 0 \\
  \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) &= m^2 
  \end{align*} \]
- Last term on LHS depends only on $l$, yet first two do not depend on $l$, so last term must be constant (and first two must add up to negative of that constant).
  \[ \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) = -m^2 \]
- This is of the form $\frac{m^2}{\sqrt{\frac{2}{m^2}}} = -\kappa x$
- This an ODE, with solution $L = \mu \cos^m \lambda + \nu \sin^m \lambda$, where $m$ is an integer.
- Going back to the first two terms, we have
  \[ \sin^2 \theta \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial R}{\partial r} \right) + \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) = m^2 
  \]
- Multiply through by $\frac{1}{\sin^2 \theta}$:
  \[ \begin{align*}
  \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( r^2 \frac{\partial R}{\partial r} \right) &= \frac{m^2}{\sin^2 \theta} \\
  \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) &= \frac{m^2}{\sin^2 \theta} \\
  \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) &= \mu + \nu \\
  \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) &= \frac{m^2}{\sin^2 \theta} \\
  \end{align*} \]
- Again, terms must be independent, so both must be constant:
  \[ \begin{align*}
  \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) &= \mu + \nu \\
  \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) &= \frac{m^2}{\sin^2 \theta} \\
  \end{align*} \]
- Finally:
  \[ \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) = \mu + \nu \\
  \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) = \frac{m^2}{\sin^2 \theta} \\
  \]
- This is known as Legendre's Equation, and has solutions of the form $T = k^n P^n \cos \lambda$, where $P^n \cos \lambda$ are the Associated Legendre Polynomials, $k^n$ are constants.

- The general solution to LaPlace's Equation, then, is:
Like any differential equation, the undetermined coefficients, in this case $C_l^m$, $C'_l^m$, $S_l^m$, $S'_l^m$ (an infinite number of them!), must be determined by boundary conditions. A few of these are "common sense" boundary conditions; the rest have to be determined by best fit of the various harmonics to the Earth's gravitational field. Since we are continually improving our knowledge of gravity, the values of these constants are being refined.

1. Since a body that is finite in three dimensions ($x, y, z$) will "look like" a point mass at infinity, the gravity must tend to $GM/r^2$ as $r$ goes to infinity, so the potential will go to $-GM/r$. This eliminates the $C_l^m, S_l^m$ terms, because they depend on $r^l$.

2. For $l = 0, m = 0$, the Legendre polynomial $P_0^m(\cos(q))$ (remember, this is a function, not a constant times $\cos(r)$) is 1, so $C_0^0$ is identically equal to $GM/r$, where $G$ is the universal constant, $M$ is the mass of the body, and $r$ is the distance from it. This term represents the "sphere" part of the potential.

3. If we set the origin at the center of mass of the body, there will be as much mass east and west of the center of mass, north and south of the center of mass, and in front and behind the center of mass. Therefore, the $l = 1, m = 0$ term must be zero, because it is asymmetrical between the northern and southern hemispheres. So, $C_1^0 = 0$. This is because $P_0^0(\cos(q)) = \cos(q)$, which is positive in the N and negative in the S (or vice versa, since $C_1^0$, if it weren't zero, could be negative).
Table C.1 Legendre Polynomials $P_l^m(\cos \theta)$ and Associated Polynomials $P_{lm}^m(\cos \theta)$. Factors in brackets convert $P_{lm}^m$ to $P_l^m$ by Eq. (C.13)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$m = 0$</th>
<th>$m = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$1$</td>
</tr>
<tr>
<td>1</td>
<td>$\cos \theta$</td>
<td>$(\sqrt{3})$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}(3\cos^2 \theta - 1)$</td>
<td>$(\sqrt{5})$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}(5\cos^2 \theta - 3\cos \theta)$</td>
<td>$(\sqrt{7})$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{8}(35\cos^4 \theta - 30\cos^2 \theta + 3)$</td>
<td>$(\sqrt{9})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$l$</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>1</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>2</td>
<td>$3\sin^2 \theta$</td>
<td>$(\sqrt{5}/12)$</td>
<td>$-$</td>
</tr>
<tr>
<td>3</td>
<td>$15\cos \theta \sin^2 \theta$</td>
<td>$(\sqrt{7}/60)$</td>
<td>$15\sin^3 \theta$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{15}{2}(7\cos^2 \theta - 1)\sin^2 \theta$</td>
<td>$(\sqrt{1}/20)$</td>
<td>$105\cos \theta \sin^3 \theta$</td>
</tr>
</tbody>
</table>


Generating Function for Legendre Polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n \text{ where } n=0, 1, 2, 3...$$

- Why Do We Care About Spherical Harmonic Analysis of Earth's Gravity?

- Examples of $P_l^m$:
  - $P_0^0 = 1$
  - $P_1^0 = \cos \theta$
  - $P_1^1 = \sin \theta$
  - $P_0^2 = \frac{1}{2}(3\cos^2 \theta - 1)$
  - $P_2^1 = 3\cos \theta \sin \theta$
  - $P_2^2 = 3 \sin^2 \theta$

- Spherical Harmonic Analysis consists of determining values for (and significance of) constants $S'$, $C'$ = 0 since $U \to 0$ as $r \to \infty$
  - for rotating Earth, might neglect $l$ dependence, i.e., allow only $m = 0$ terms:

$$U = -\frac{1}{\tilde{\alpha}} \sum_{l=0}^{\infty} C_l^0 \left(\frac{\tilde{\alpha}}{r}\right)^{l+1} P_l^0(\cos \theta) \text{ where } P_l^0(\cos \theta) \text{ are Legendre polynomials}$$

- or, for convenience
$U = -\frac{GM}{r} \left( J_0 P_0 - J_1 \frac{\alpha}{r} P_1 (\cos \theta) - J_2 \left( \frac{\alpha}{r} \right)^2 P_2 (\cos \theta) + ... \right)$

- $J_0 = 1$ because as $r \to \infty$, $U \to -\frac{GM}{r}$
- $J_1 = 0$ if we pick origin to be center of mass
- $J_n = 0$, $n$ odd, if equator is plane of symmetry
- $U = -\frac{GM}{r} \left( 1 - \frac{1}{2} \left( \frac{\alpha}{r} \right)^2 J_2 (3 \cos^2 \theta - 1) + ... \right)$

- Values determined by satellite:
  - $J_0 = 1$
  - $J_1 = 0$
  - $J_2 = 1,082.635 \pm 0.011 \cdot 10^{-6}$ (oblateness)
  - $J_3 = -2.531 \pm 0.007 \cdot 10^{-5}$ (pear-shapedness)
  - $J_4 = -1.600 \pm 0.012 \cdot 10^{-5}$
  - $J_5 = -0.246 \pm 0.9 \cdot 10^{-5}$
- Measurements of Earth's gravity field show that the biggest effect is due to Earth's rotation and bulge

This is the differential equation for the Simple Harmonic Oscillator (SHO), or a mass on a spring:

$$m \frac{\partial^2 x}{\partial t^2} = -kx$$

where $t$ is time, $x$ is displacement of the spring, $m$ is the mass and $k$ is the spring constant. The general solution to this equation is:

$$x = A \cos \left( \sqrt{\frac{k}{m}} t \right) + B \sin \left( \sqrt{\frac{k}{m}} t \right)$$

The undetermined coefficients $A$ and $B$ are determined by initial conditions (think of them as boundary conditions in time), namely the position $x$, and velocity $v$, of the mass, when $t = 0$. 