GRACE time-variable gravity field recovery using an improved energy balance approach

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SUMMARY
A new approach based on energy conservation principle for satellite gravimetry mission has been developed and yields more accurate estimation of in situ geopotential difference observables using K-band ranging (KBR) measurements from the Gravity Recovery and Climate Experiment (GRACE) twin-satellite mission. This new approach preserves more gravity information sensed by KBR range-rate measurements and reduces orbit error as compared to previous energy balance methods. Results from analysis of 11 yr of GRACE data indicated that the resulting geopotential difference estimates agree well with predicted values from official Level 2 solutions: with much higher correlation at 0.9, as compared to 0.5–0.8 reported by previous published energy balance studies. We demonstrate that our approach produced a comparable time-variable gravity solution with the Level 2 solutions. The regional GRACE temporal gravity solutions over Greenland reveals that a substantially higher temporal resolution is achievable at 10-d sampling as compared to the official monthly solutions, but without the compromise of spatial resolution, nor the need to use regularization or post-processing.

Key words: Satellite geodesy; Geopotential theory; Time variable gravity.

1 INTRODUCTION
Launched in March 2002, the Gravity Recovery and Climate Experiment (GRACE) mission (Tapley et al. 2004a) has been mapping Earth’s time-variable gravity field for more than a decade and achieving remarkable and even transformative scientific advances (Cazenave & Chen 2010). From the data collected by the K-band ranging (KBR) low–low satellite-to-satellite tracking (SST) as well as the high-low GPS tracking, monthly mean gravity field models, known as the Level-2 data products, have been routinely estimated by the University of Texas Center for Space Research (CSR), GeoForschungsZentrum (GFZ) Helmholtz-Centre Potsdam German Research Centre for Geosciences, NASA’s Jet Propulsion Laboratory (JPL) and others. The estimation approach used by the above three agencies and others (e.g. Luthcke et al. 2006; Bruinsma et al. 2010) to generate these solutions is the so-called dynamic method based on the dynamical orbit determination and geophysical parameter recovery principle (Tapley et al. 2004b). Dynamic method treats both KBR and GPS tracking as observations but with different weights, and simultaneously estimates for the state parameters including the gravity coefficients, orbits and others in a least-squares solution. Because of the non-linear relationship between observations and the state parameters, linearization of both the dynamical equation of motion and the observation-state equations are required during the estimation process. Besides the conventional dynamic method, various alternative approaches have also been proposed and implemented, such as mascon approach (Rowlands et al. 2005, 2010), short-arc approach (Mayer-Gürr et al. 2007; Kurtenbach et al. 2009), celestial mechanics approach (Meyer et al. 2012), acceleration approach (Ditmar & van Eck van der Sluijs 2004; Chen et al. 2008; Liu et al. 2010) and the energy balance approach (Jekeli 1999; Han et al. 2006; Ramillien et al. 2011; Tangdamrongsub et al. 2012). The last approach is the focus of this paper.

Energy balance approach, also known as energy integral approach, can be traced back to the 1960s (e.g. Bjerhammar 1969), in the early era of satellite geodesy. The basic idea of this approach is to explore the possibility of applying the principle of energy conservation, that is the constant sum of kinematic energy and potential energy, to the SST data for direct measuring of Earth’s gravity field. The concept was investigated again by Jekeli (1999) at the onset of the Decade of Geopotential Missions, and he developed the first practical formulation to explicitly express the relationship between geopotential and satellite data in inertial frame (later called the energy equation), with conceived application for the forthcoming satellite gravimetry missions, Challenging Minisatellite Payload (CHAMP) and GRACE. Shortly after, Visser et al. (2003) similarly derived the energy equation but in the Earth-fixed frame. Since then, a renewed interest of using energy balance approach to estimate Earth’s static and time-variable gravity field was aroused during the last decade, especially for application using the data from
Earth satellite gravimetry missions, such as CHAMP (e.g. Han et al. 2002; Gerlach et al. 2003; Badura et al. 2006), GRACE (e.g. Han et al. 2006; Ramillien et al. 2011; Tangdamrongsub et al. 2012) and Gravity Field and Steady-State Ocean Circulation Explorer (GOCE; e.g. Pail et al. 2011).

One of the major advantages of energy balance approach is that it can be utilized to estimate the in situ geopotential observables (for a single satellite) or geopotential difference observables (for a pair of satellites), which means that the geopotential or geopotential differences can be computed at the satellite altitude, and then used to solve for the Earth’s gravity field. Similarly, acceleration approach (Ditmar & van Eck van der Sluijs 2004) can also generate in situ observables, but in the form of acceleration. In contrast, conventional dynamic method normally cannot provide this kind of in situ observables, where the geometric measurements, that is KBR and GPS tracking, have to be applied to directly solve gravity field, that is Stokes coefficients. The in situ geopotential observables, as a quantity with explicit geophysical interpretation, can serve as an intermediate product between the satellite measurements and final gravity solutions, since the estimation procedure is more efficient because of the linear relationship between the observables and gravity coefficients. More importantly, the in situ geopotential difference observables would greatly benefit the time-variable gravity recovery missions, such as GRACE, since the epoch-wise observables can support flexible spatial and temporal resolutions, leading to regional solutions with possibly retrieving more local gravity information (Han et al. 2005; Schmidt et al. 2006, 2008; Tangdamrongsub et al. 2012).

However, appropriate application of energy balance approach on GRACE-type mission data for highly accurate geopotential estimation is still a demanding task. One of the most challenging problems is how to efficiently extract the gravity signal sensed by the essential measurements from SST, that is KBR range-rate measurements, which the energy equation does not explicitly contain. Previous researchers attempted to adjust range-rate and orbit data simultaneously via a non-linear least-squares estimation with either fixed constraints (Han et al. 2006) or via stochastic constraints (Tangdamrongsub et al. 2012). The use of constrained least-squares adjustment, though straightforward, is still a compromise between the very high-precision range-rate data and the relative low-precision orbit data, which may tend to distort the estimation of in situ geopotential observables caused by errors including orbit error. The orbit error, inherited from the chosen reference orbit, would contaminate the resulting gravity estimation especially at the low-frequency band (Ditmar et al. 2012). In addition, our recent study (Guo et al. 2015) has demonstrated that the previous formulation of the energy equation may contain a non-negligible approximation, which could sometimes overwhelm the time-variable gravity signal also at the low-frequency band. These issues limit the application of the earlier developed in situ geopotential differences only to regional gravity analysis, that is at the high-frequency band, and arguably over regions with large temporal gravity field signals. As a result, large-scale gravity field inversion, including global gravity solution, has not been fully exploited based on previous energy approach.

The primary purpose of this study is to overcome these limitations by employing an improved energy balance approach to obtain a more accurate estimation of in situ geopotential difference observables, with the aim to preserve both the low- and high-frequency gravity signal and consequently yield a full scale, that is both regional and global, gravity inversion. To achieve this goal, we develop a novel formulation, called the alignment equation, to incorporate range-rate observations into energy equation, together with a method to reconstruct the related reference orbit. In addition, a more rigorous formulation of energy equation (Guo et al. 2015) is applied to model the in situ geopotential difference observables, which is requisite for the reduction of the GRACE measurements for gravity field inversion.

The outline of this paper is as follows: Section 2 starts with a brief description of the general idea of energy balance formalism, followed by a detailed description of the methodology about our improved approach. Section 3 presents the numerical results using the new energy approach, on the validation of the accuracy of the resulting improved in situ geopotential difference observables, and on using the observables to demonstrate monthly and 10-d temporal gravity recovery. Finally, conclusions and discussions are summarized in Section 4.

2 METHODOLOGY

The orbit data, both positions and velocities, are dominated by the gravitational perturbations and other forces, and thus can be regarded as observations and used for gravity recovery after proper reduction of other forces, which is the basic concept of the energy balance approach. The energy equation, a mathematical expression of this concept, can be formulated in Earth-centred inertial frame (Jekeli 1999) for a single satellite as

\[ V = \frac{1}{2} |r|^2 + \int_0^t \frac{\partial V}{\partial t} dt - \int_0^t f \cdot \dot{r} dt - E_0, \]

(1)

where \( V \) is the total gravitational potential (for unit mass), \( r \) (implicit in \( V \) and \( f \)) is the orbit position and velocity in inertial frame, \( f \) is the non-conservative force, \( \int_0^t \left( \partial V / \partial t \right) dt \) is the so-called potential rotation term, and \( E_0 \) is an integral constant. The total gravitational potential \( V \) can be decomposed into two parts \( V = V^E + V^R \), where \( V^E \) is the geopotential, including the Earth’s mean, secular, seasonal and other variable components, and \( V^R \) is the residual gravitational potential, mostly from the high-frequency (e.g. semi-diurnal and diurnal) variable geopotential, such as tides. If we assume the residual gravitational potential \( V^R \) can be reduced or corrected using a priori model, and also non-conservative force \( f \) can be measured by an on-board accelerometer, we arrive at the complete formulation of energy equation for estimating geopotential \( V^E \), from a single satellite, such as CHAMP, which can be expressed as

\[ V^E = \frac{1}{2} |r|^2 + \int_0^t \frac{\partial V}{\partial t} dt - \int_0^t f \cdot \dot{r} dt - V^R - E_0. \]

(2)

And for estimating geopotential difference from a pair of satellites, such as GRACE, the formulation is simply the subtraction between the equations of two single satellites:

\[ V^E_{12} = V^E_2 - V^E_1 \]

\[ = \frac{1}{2} |r_{12}|^2 + \dot{r}_1 \cdot \dot{r}_{12} + \int_0^t \frac{\partial V_{12}}{\partial t} dt 
\]

\[ - \int_0^t \left( f_2 \cdot \dot{r}_2 - f_1 \cdot \dot{r}_1 \right) dt - V^R_{12} - E_0_{12}, \]

(3)

where the subscripts represent the two satellites (‘1’, ‘2’) and their difference (‘12’). It is worth mentioning that the orbit data \( r \) and \( f \) in both formulations are normally regarded as observables, which are usually from a precomputed reference orbit using high-low GPS tracking data.
2.1 A novel method to incorporate range-rate measurements into energy equation

Application of energy balance approach on GRACE-type mission is much more challenging than CHAMP-type mission because energy eq. (3) is unable to explicitly contain the tracking measurements from the low–low SST system, that is range-rate measurements from KBR system in the case of GRACE. Previous studies of energy balance approach usually treat range-rate measurements as a kind of redundancy observation, and adjust them simultaneously with orbit data along the orbit, by a non-linear least-squares estimation, where the energy equation is treated as either a fixed (Han et al. 2006) or stochastic (Tangdamrongsub et al. 2012) constraint. The estimates would be the six intersatellite orbit state components with some other empirical parameters. However, we know that the uncertainty of GRACE orbits is around 1–2 cm in positions and 10–20 μm s⁻¹ in velocities (only for dynamic orbit; for kinematic orbit the uncertainty in velocities is even worse; Kang et al. 2006), whereas the range-rate measurements have a much lower uncertainty of about 0.2 μm s⁻¹ (Loomis et al. 2012) at high-frequency band. The use of constrained least-squares adjustment in energy balance approach may be able to extract some information from range-rate measurement, but it’s still a compromise between high-precision data and low-precision data, as the (unknown) systematic error, for example from orbit errors, would inevitably affect the solved parameters, and subsequently bias the estimation of geopotential difference observables.

Therefore, in this study we present a new, alternative method to adjust intersatellite orbit state components using range-rate measurements. The motivation is based on a simple fact, which is that the range-rate measurements cannot be sensitive to all the intersatellite orbit components. Previous study by Rowlands et al. (2002) has already shown that, among all the intersatellite components, the relative velocity pitch is the most sensitive to range-rate measurements, and also one of the most important components for gravity recovery (Luthcke et al. 2006). The other two important components are relative velocity magnitude and relative position pitch, but these are much less sensitive to range-rate measurements compared to relative velocity pitch. Based on this fact, we aim to develop a method to use range-rate measurements to only adjust the most sensitive component, that is relative velocity pitch, and adopt the other less sensitive or insensitive components to be provided by the reference orbits.

The method is accomplished through a simple equation as follows:

\[ \hat{\rho}_{12} = \hat{\rho} \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|} + \sqrt{|\mathbf{r}_{12}|^2 - \hat{\rho}^2} \frac{\mathbf{r}_{12} \times \hat{\mathbf{r}}_{12} \times \mathbf{r}_{12}}{|\mathbf{r}_{12} \times \hat{\mathbf{r}}_{12} \times \mathbf{r}_{12}|} \]  

(4)

which is referred to as alignment equation throughout this paper. Here \( \hat{\rho} \) represents the range-rate measurement, \( \mathbf{r}_{12} \) and \( \hat{\mathbf{r}}_{12} \) are the relative position and velocity vector from reference orbit, \( \mathbf{r}_{12} \times \hat{\mathbf{r}}_{12} \times \mathbf{r}_{12} \) is the vector triple product between them, and \( \hat{\mathbf{r}}_{12} \) represents the new relative velocity vector. As we can see, the new relative velocity vector \( \hat{\mathbf{r}}_{12} \) would be equal to the original vector \( \hat{\mathbf{r}}_{12} \) if there were no additional range-rate observation, that is \( \hat{\rho} = \hat{\rho}_{12} \cdot \mathbf{r}_{12}/|\mathbf{r}_{12}| \). In that case, the alignment equation would degrade to an identical equation, which represents an exact geometric relationship between relative velocity direction vector and range-rate measurement. That means the alignment equation itself does not contain any approximation.

Once we have an independent and more accurate measuring of relative range-rate, such as the case of GRACE, then the new relative velocity vector \( \hat{\mathbf{r}}_{12} \) would become more accurate compared to the original vector \( \mathbf{r}_{12} \) because the pitch angle of relative velocity vector has been constrained by, or we can say, aligned to the range-rate. The term ‘alignment’ actually means the relative velocity pitch, the most sensitive intersatellite parameter to range-rate and most important parameter for gravity recovery, has been aligned to the range-rate measurement through the equation. The reason can be further explained as follows.

The alignment equation decomposes relative velocity vectors into two components. Fig. 1 illustrates such decomposition by showing the simple geometric configuration of GRACE constellation, where Fig. 1(a) shows the absolute position and velocity vector for GRACE satellites with respect to the Earth’s centre of mass (CM), and Fig. 1(b) shows the decomposition of velocity vector. As shown in Fig. 1(b), the intersatellite velocity is decomposed into two orthogonal directions. One is along the line-of-sight (LOS) direction, where unit vector is \( \mathbf{n}_1 = \mathbf{r}_{12}/|\mathbf{r}_{12}| \), and the correspondent projection is \( \hat{\rho} \). The other is orthogonal to the LOS direction and is in the plane containing relative position vector and velocity vector, where the unit vector is \( \mathbf{n}_2 = (\mathbf{r}_{12} \times \hat{\mathbf{r}}_{12} \times \mathbf{r}_{12})/|\mathbf{r}_{12} \times \hat{\mathbf{r}}_{12} \times \mathbf{r}_{12}| \), with the correspondent projection of \( \sqrt{|\mathbf{n}_{12}|^2 - \hat{\rho}^2} \) in order to maintain the same magnitude of the intersatellite velocity.

Again, our goal here is to use high accurate measurement, that is the range-rate measured by the KBR, to adjust the most sensitive intersatellite parameter, that is the relative velocity pitch. In another words, range-rate should be used to replace the relative velocity pitch component and form a new ‘pitch-free’ relative velocity vector. That is exactly what the alignment equation represents. Under the decomposition as eq. (4), the computation of \( \mathbf{n}_{12} \) only requires four intersatellite quantities, which are range-rate \( \hat{\rho} \), relative velocity magnitude \( |\mathbf{F}_{12}| \), LOS direction unit vector \( \mathbf{n}_1 \), and direction unit vector \( \mathbf{n}_2 \) that is always perpendicular to \( \mathbf{n}_1 \), and in the plane of intersatellite position and velocity. Among the four quantities, two of them, \( \hat{\rho} \) and \( \mathbf{n}_1 \), are totally independent of the velocity component (as well as the position magnitude), and \( |\mathbf{F}_{12}| \) is only dependent on the velocity magnitude. The last one, unit vector \( \mathbf{n}_2 \), does rely on the velocity direction, but only the position magnitude, and \( |\mathbf{F}_{12}| \) is only dependent on the velocity magnitude. Therefore, by using the alignment equation, the components that are needed from reference orbit are only relative velocity magnitude, relative position direction and relative velocity yaw, but not relative velocity pitch. The effect of relative velocity pitch from less accurate reference orbit is thus totally eliminated. The only contribution of the resulting relative velocity pitch is from range-rate, and
therefore the most sensitive component to intersatellite observation, has been fully constrained by range-rate measurement through the alignment equation.

After applying the alignment equation, the new relative velocity vector \( \vec{r}_{12} \) would subsequently be used as the input of energy eq. (3). In the next step, we expect to determine geopotential difference observables solely from range-rate measurements, and meanwhile minimize the direct effect from the reference orbit to the estimates, which will be discussed in the next subsection.

### 2.2 Reconstruction of the reference orbit

The reference orbit is another critical input to the energy eq. (3) in addition to the range-rate measurements. Generally there exist three different choices of reference orbits, namely the kinematic, reduced-dynamic or dynamic orbit, for the case of GRACE. Since range-rate should dominate the time-variable gravity information, the geopotential difference estimates do not rely much on the choice of the reference orbit. Therefore, it is possible to choose dynamic or reduced-dynamic orbit as the reference orbit for GRACE gravity field recovery, as long as the range-rate measurements are appropriately used to correct or adjust the orbit data. In practice, various reference orbit data have indeed been implemented for GRACE real data analysis (Han et al. 2006; Tangdamrongsub et al. 2012).

In this study, we choose to adopt a pure dynamic orbit as the reference orbit. However instead of computing a dynamic orbit directly from GPS observations, we use an alternative method to reconstruct the pure dynamic orbit from existing orbit data products. The similar technique has been used for previous studies on GRACE (Liu et al. 2010) and GOCE (Yi 2012). The idea is to treat the available orbit coordinates as pseudo observations, and estimate a pure dynamic orbit by fitting the orbit coordinates with respect to a complete reference model, via least-squares adjustment. Meanwhile, the accelerometer data are also simultaneously calibrated with respect to the pure dynamic orbit. The reference models we used in this study are identical to the models used by GFZ for solving the official GRACE Level-2 (L2) product Release 05 (RL05; Biancale & Bode 2006; Petit & Luzum 2010; Dahle et al. 2012; Mayer-Gürr et al. 2012). We also adopt a similar strategy for calibrating accelerometer data, which is to estimate daily biases, and monthly scale factors. The accelerometer data from GRACE Level 1B (L1B) ACC1B product, together with orientation data from L1B SCA1B product, are used to model the non-gravitational forces. The GRACE L1B data can be downloaded via http://podaac.jpl.nasa.gov/GRACE.

As for the input of the reconstruction of the pure dynamic orbit, we have tested three different highly accurate scientific orbit products from independent institutes, which are kinematic orbit product from National Central University, Taiwan (T. Tseng, personal communication, 2014), kinematic orbit product from University of Bern (A. Jäggi, personal communication, 2014), and reduced-dynamic orbit from JPL, that is the GRACE L1B GVNV product. We found the difference between the resulting reconstructed pure dynamic orbits using these orbit products negligible. For example of one satellite on 2009 January 25, the direct orbit position difference between two products from University of Bern and JPL is about 2.48 cm (rms), and after reconstruction, the resulting orbit difference is reduced to 0.76 cm (rms). More importantly, the subsequent geopotential difference observables are also not sensitive to the input orbit product because of the reconstruction process. For the resulting geopotential differences on the same day, the difference between the two estimates based on the two orbit products is only \( 1.1 \times 10^{-4} \text{ m}^2\text{s}^{-2} \) (rms), which is just above the error level of the formulation of energy equation (cf. Fig. 2b). Therefore, we simply choose the L1B GNV1B product as the input orbit data to reconstruct the pure dynamic orbit since it is readily available with other L1B products.

The purpose of using a reconstructed pure dynamic orbit is to minimize the effect of mismodelled or unmodelled errors from the pre-computed dynamic or reduced-dynamic orbit, and to suppress the random error from the kinematic orbit. When a pure dynamic orbit computed from a priori gravity model is used as the only input for energy equation, the output from both eqs (2) and (3) would be inevitably reduced to the same a priori gravity model. That means no new geopotential information would be obtained from energy equation if range-rate data were absent. However, once the pure dynamic orbit has been aligned through the alignment equation by including the rang-rate measurements, the updated orbit \( \vec{r}_{12} \) should contain the new time-variable gravity information propagated solely from the range-rate measurement, which would be revealed by the energy equation afterward. One may argue that this process may also eliminate the possible contribution from GPS tracking data to gravity estimation, but considering the much higher \((\sim 50 \text{ times})\) noise level (both high frequency and low frequency) in the orbit data as compared to the accuracy of the KBP range-rate data, we believe it is a reasonable trade-off. Besides, it is worth pointing out the same strategy is also used by Luthcke et al. (2006). They applied the traditional dynamic method to solve monthly solution, but also solely from range-rate measurements, and achieved
comparable GRACE solutions as compared to the official GRACE Level 2 data products.

2.3 Reformulation of energy equation

When energy eq. (3) is applied to compute the geopotential difference, all the quantities and terms on the right-hand side can be computed from data or based on reference models, except one term, \( \int_0^t \left( \frac{\partial V}{\partial t} \right) dt \), the so-called ‘potential rotation term’. In the previous formulation developed by Jekeli (1999)

\[
V^E \approx \frac{1}{2} |\mathbf{r}|^2 - \omega (x\dot{y} - y\dot{x}) - \int_0^t \mathbf{f} \cdot d\mathbf{t} - V^R - E^0,
\]

the potential rotation term was approximated as \( \int_0^t \left( \frac{\partial V}{\partial t} \right) dt \approx -\omega (x\dot{y} - y\dot{x}) \), where \( x \) and \( y \) represent the first and second component of the position vector, and \( \omega \) is the nominal mean Earth’s angular velocity. However, Guo et al. (2015) have already demonstrated via simulation that this approximation of potential rotation term exceeds the precision of GRACE observation, and suggested a more accurate formulation of energy equation as follows:

\[
V^E \approx \frac{1}{2} |\mathbf{r}|^2 - \mathbf{w} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) - \int_0^t \mathbf{a} \cdot (\dot{\mathbf{r}} - \mathbf{w} \times \mathbf{r}) dt - E^0,
\]

where \( \mathbf{a} = \nabla V^R + \mathbf{f} \) is the acceleration of both residual geopotential acceleration and non-conservative acceleration, and \( \mathbf{w} \) is Earth’s angular velocity of Earth-fixed frame relative to the inertial frame, with coordinates in the inertial frame. The third term of the right-hand sides can be numerically integrated. The detailed derivation can be found in the Appendix, where we also demonstrate the equivalence of the energy equations in both inertial frame and Earth-fixed frame (Jäggi et al. 2008; Wang et al. 2012). Thus there is no difference no matter in which frame the energy equation is used for the energy balance method.

The Appendix also shows that two approximations are included in eq. (6). One is to assume \( V^E \) to be static during the integral limits from \( t_0 \) to \( t \), which is consistent with the normal GRACE convention that is estimating a mean gravity field during a certain time interval. The other one is to assume the rates of Earth’s angular velocity vector is zero, that is \( \dot{\mathbf{w}} = 0 \), which is negligible compared to measurement noise level of GRACE (Guo et al. 2015). In contrast, eq. (5) contains too many rough approximations that would pollute the signal level of GRACE. In order to check the accuracy of the two formulations, a closed-loop validation is performed. We first simulate pure dynamic orbits for two satellites as the input to the energy equation. As we mentioned in the last subsection, using a pure dynamic orbit data as the input of energy equation should reduce the estimates to the \( a \) priori gravity field. If we define residual geopotential difference observables as: \( \Delta V^E_{12} = V^E_{12} - V^E_{12}^{\text{prior}} \), that is the difference between the estimated values using energy equation and the predicted values using the \( a \) priori gravity model, then theoretically the residual should be reduced to zero, and therefore the non-zero residual would reveal the approximation level for each formulation of energy equation.

The residuals based on eqs (5) and (6) are presented in Fig. 2, for an integration period of 1 d. Fig. 2(a) shows the residuals based on both equations, and we can see that the one based on eq. (6) is reduced to almost zero, but the one based on eq. (5) contains a relative large non-zero residual with a dominate 2 cycle-per-revolution (CPR) error, which is caused by the approximation primarily from the potential rotational term. A zoomed-in view of the error from eq. (6) is presented in Fig. 2(b), and the peak-to-peak amplitude is less than \( 1 \times 10^{-4} \) m² s⁻², which is definitely negligible for current GRACE measurement accuracy and probably also for GRACE follow-on measurement accuracy in the future (Loomis et al. 2012). More detailed numerical comparison can be found in Guo et al. (2015). The error level based on eq. (5) is about 0.02 m² s⁻² from peak to peak, which is larger than the signal level from the time-variable gravity field (see next Section on the resulting geopotential difference estimates using real data). All these errors caused by the approximation would surely corrupt the geopotential difference estimates. It seems that the dominant errors have the frequency close to 2 CPR, which might be removed by an additional 2 CPR parameters (Han et al. 2006; Tangdamrongsub et al. 2012) or even more parameters (Ramillien et al. 2011), but actually the errors contain much more high-frequency constituents (e.g. from ocean tides), which empirical parameters cannot fully absorb these errors. Also we know that 2 and higher CPR empirical parameters would heavily contaminate gravity signal, especially the zonal geopotential coefficients. The studies based on traditional orbit dynamic approach also indicate that the empirical parameterization should be no more than 1 CPR (Tapley et al. 2004a) or even less parameters (Luthcke et al. 2006), which is for the purpose to mitigate the systematic error of range-rate data, and also to better retain the time-variable geopotential signal. Therefore, we conclude that in order to fully exploit the precision of GRACE data, it is requisite to choose eq. (6) as the practical formulation of energy equation.

3 Results and analysis

3.1 Geopotential difference estimates

After the reduction of non-conservative acceleration and other tidal potential terms, each geometric observation from GRACE can be directly linked to a geophysical quantity, that is geopotential difference between the two positions of the twin satellites, and therefore energy balance approach could provide a unique ability to extend our conventional knowledge about both data processing and results interpretation of GRACE. The geopotential difference estimates are able to directly sense the gravity information, without losing any high-frequency resolution, since each estimate is computed straight from range-rate measurement for each epoch. Because of that, the geopotential difference estimates could not only be used for both global and regional gravity recovery but also be regarded as an \( in \) \( situ \) gravity representation without downward continuation (Han et al. 2006). Therefore, an accurate estimation of geopotential difference observables is the key issue for energy balance approach and is the most critical step for the subsequent temporal gravity inversion.

Using the methods described in the previous section, geopotential differences are estimated for each day from 2003 to 2013. All the input data are from GRACE L1B data products, including GNV1B orbit data that are used to reconstruct the pure dynamic orbit and estimate the daily accelerometer calibration parameters. Range-rate data from KBR1B product are then included to correct the velocity components of the reconstructed orbit via the alignment eq. (4). Next, energy eq. (3) is applied according to the formulation of eq. (6), to compute the geopotential difference observables.

The accelerometer measures the non-conservative force in three orthogonal directions of the Science Reference Frame (SRF), which can be transformed into inertial frame using the quaternions from the SCA1B product measured by the Star Camera Assembly. Approximately, \( X \) direction is along roll axis in the anti-flight and
Figure 3. Geopotential difference estimates (with mean gravity field model removed). (a) Ground track of 2003 July 17 with two ascending profiles highlighted with colour representing the values of the estimates. (b) Highlighted profile approximately along 60°W longitude mostly above the land area, with predicted values from GRACE L2 solutions (CSR RL05, GFZ RL05a and JPL RL05). (c) The same but for highlighted profile approximately along 160°W longitude mostly above the ocean area. (d) Time-series of estimates for the day of 2006 May 1 with predicted values from CSR RL05 solution.

In-flight directions for the leading and trailing satellites, respectively, Z direction is along the yaw axis and points to nadir and Y direction is along pitch axis and forms a right-handed triad with X and Z. For GRACE satellite A, the means of the estimated accelerometer biases are −1.1825, 29.4500 and −0.5527 µ s⁻², and the rms values about the means are 0.0388, 1.1645 and 0.0687 µ s⁻², for the X-, Y- and Z-axes, respectively. For GRACE satellite B, the means of the biases are −0.5810, 10.9709 and −0.7616 µ s⁻², and the rms values about means are 0.0211, 0.8170 and 0.1530 µ s⁻². For the scale factors of satellite A, the means are 0.9493, 0.9448 and 0.9520, and the rms values about means are 0.0769, 0.2156 and 0.1962, and for satellite B, the means are 0.9358, 0.9494 and 0.9572, and the rms values about means are 0.0651, 0.2065 and 0.1742. For the systematic errors in the geopotential differences, a number of empirical parameters are estimated and removed from the geopotential difference observables. These include daily bias, rate and 1 CPR parameters for every orbital revolution, which absorb not only the systematic errors from range-rate data, but also the integral constants from the energy equation.

In Fig. 3(a), we highlight two ascending profiles of geopotential difference observables on a global map of the daily ground tracks of 2003 July 17. The geographical coordinates of each estimate on the map are assigned to the middle point of two satellites and the colour represents its value with a mean reference gravity field GIF48 (Ries et al. 2011) removed. In Figs 3(b) and (c), the two profiles of estimates are, respectively, illustrated with respect to latitude and compared to the predicted values using GRACE L2 solutions (from CSR RL05, GFZ RL05a and JPL RL05, truncated to degree and order 60 with the same mean field removed) along the same profiles. We can clearly see that the geopotential differences estimates using our approach are similar to the predicted values from the three official L2 solutions. Of course, the three series of predicted values look smooth since they are simply computed from existing models with a gravity field up to degree and order 60 only, while the series of estimates is noisier because it is computed from range-rate measurements directly.

Fig. 3(b) illustrates the profile approximately along 60°W longitude, mostly above the rough land area, and Fig. 3(c) is for the profile approximately along 160°W longitude, mostly above the flat ocean area. With the ascending of the satellite pair, we can observe the estimates in Fig. 3(b) reveal the geopotential difference variation successively caused by West Antarctica, Amazon Basin, Hudson Bay and North Greenland, and the estimates in Fig. 3(c) mostly cover the Pacific Ocean. As we can see, the surface gravity change may be directly inferred just from the geopotential difference profile. Since here we define the difference as the following satellite subtracting the leading satellite, when the satellite pair passes by a negative gravity anomaly on ground, the geopotential difference profile will exhibit increasing values first and decreasing values next, and vice versa for a positive anomaly. For example, in Fig. 3(b), from about 30°N to North Pole, we can observe an increase-decrease-increase fluctuation of geopotential differences, meaning there should be a negative gravity anomaly followed by a positive anomaly compared to the epoch of the mean reference field (2007 January 1), which corresponds to the glacial isostatic adjustment (GIA) signal (negative) in the Hudson Bay and Greenland ice sheet ablation signal (positive).
In Fig. 3(d), we show a profile as a time-series for a different day (2006 May 1) with predicted time-series from CSR L2 RL05 solution for comparison. Again, the time-series of the estimates seems very close to the predicted values from CSR solution. In order to compare with previous study, we adopt the same method from Han et al. (2006) to compute the correlation coefficient of the time-series between the (smoothed) estimates and the predicted values. The resulting correlation coefficient is about 0.91 for that particular day. The rms of the difference between the (smoothed) estimates and the predicted values is about $8.4 \times 10^{-4}$ m$^2$ s$^{-2}$, and for comparison, the rms of the values themselves is about 0.002 m$^2$ s$^{-2}$. As for all the estimates from 2003 to 2013, the average value of the daily correlation coefficients is over 0.9, which is much higher than correlations of 0.5–0.8 reported in previous study by Han et al. (2006).

3.2 Global gravity solutions using geopotential difference observables

As an in situ observation type, geopotential differences have been widely used for gravity recovery, especially for regional gravity recovery. However, the global gravity recovery using geopotential differences is not commonly used, even if it is more straightforward because of the linear relationship between geopotential differences and Stokes coefficients. In this subsection, we explored the possibility of global gravity recovery based on the geopotential difference estimates from our improved approach.

Similar to the convention of official GRACE L2 product, we also produce monthly mean solutions for each calendar month. First, similar to Fig. 3(a), we plot all the accumulated geopotential difference estimates for an example month of July 2003 on the global map in Fig. 4(a). The data from descending passes are presented with an additional minus sign so they would not look opposite to the data from ascending passes over the same region of the global map. From Fig. 4(a), we can see that some regions with large gravity variation are manifested in the global map, including not only the highlighted regions in Fig. 3(a) but also some other regions like Alaska (glacier melting), Congo Basin (wet season), and Scandinavia (GIA).

For comparison, the other three figures in the left-hand column of Fig. 4 show the predicted values at the same geographical location from three GRACE L2 solutions completed to spherical harmonic degree and order 60 (Fig. 4c for CSR RL05, Fig. 4e for GFZ RL05a and Fig. 4g for JPL RL05). The most significant discrepancy between Fig. 4(a) and other three figures in the left (Figs 4c, 4e and 4g) is that Fig. 4(a) apparently contains measurement noise which is inherited from each range-rate measurement, while other three figures are only predicted from a truncated gravity field model with resolution of up to degree 60. That shows the key contribution of using energy balance approach, that is directly connecting the geometry measurements (range-rate) to the geophysical quantities (geopotential difference). Therefore, even though we plot the whole month estimates in the same global map, Fig. 4(a) still preserve the in situ geopotential change for each epoch within a month, that is submonthly information, but Figs 4(c), (e) and (g) can only show the predicted values from a monthly mean (static) gravity field.

Next, we use the accumulated estimates to produce a monthly global solution. The relation of geopotential difference $V_{12}^E$ and Stokes coefficients ($\bar{C}_{nm}$ and $\bar{S}_{nm}$) can be expressed as

$$V_{12}^E = \frac{GM}{R} \sum_{n=2}^{n_{max}} \sum_{m=0}^{n} (\alpha_{nm} \bar{C}_{nm} + \beta_{nm} \bar{S}_{nm}),$$

where $GM$ is the geocentric gravitational constant and $R$ is Earth’s radius, $n$ and $m$ are degree and order, respectively, and $n_{max}$ is the maximum degree, that is 60 in this study. Here we exclude degree 0 and degree 1 coefficients, as GRACE range rate measurements are insensitive to these parameters. The coefficients $\alpha_{nm}$ and $\beta_{nm}$ are defined as

$$\begin{align*}
\sigma_{nm} &= \left( \frac{R}{r_2} \right)^{n+1} \bar{P}_{nm}(\cos \theta_2) \left\{ \cos (m \lambda_2) \sin (m \lambda_2) \right\} \\
\beta_{nm} &= -\left( \frac{R}{r_1} \right)^{n+1} \bar{P}_{nm}(\cos \theta_1) \left\{ \cos (m \lambda_1) \sin (m \lambda_1) \right\},
\end{align*}$$

where ($r_1$, $\theta_1$, $\lambda_1$) and ($r_2$, $\theta_2$, $\lambda_2$) are denoted as the spherical coordinates of the two satellites in Earth-fixed reference system, and $\bar{P}_{nm}$ is the fully normalized Legendre function. Based on the least-squares principle, the solution of the unknowns ($\bar{C}_{nm}$ and $\bar{S}_{nm}$) can be easily solved from a large number of observations ($V_{12}^E$).

The recovered monthly gravity solution is shown in Fig. 4(b) in terms of geoid undulation, using all the geopotential differences from Fig. 4(a). For comparison, the geoid maps from other three L2 solutions for the month of July 2003 are also shown in the right column of Fig. 4 (Fig. 4d for CSR RL05, Fig. 4f for GFZ RL05a and Fig. 4h for JPL RL05). Here we did not apply any post-processing techniques except that we replace the $C_{20}$ coefficients using the values obtained from satellite laser ranging (Cheng et al. 2013). Although the input geopotential differences in the left-hand column of Fig. 4(a) seem to agree with the predicted value, this does not necessarily guarantee the recovered solution should be similar as well. In fact, the high frequency signal and noise in Fig. 4(a) could still leak into the low frequency band during the inversion process. Therefore, after downward continuation from satellite altitude to Earth surface, enlarged noise, that is north-to-south stripes, should be expected in the gravity solution, which can be seen from Fig. 4(b). Compared Fig. 4(b) and other three figures in the right-hand column of Fig. 4, the geoid undulation map from our solution (OSU) has fewer stripes than the JPL or the GFZ solution, but slightly more stripes than the CSR solution, for this particular month. The comparison of Power Spectral Density (PSD) is shown in Fig. 5(a). We can see at lower degree (below degree 15), our PSD (OSU) matches other three PSDs very well, which means our solution contains highly consistent time-variable gravity signal. At higher degree (above degree 15), our PSD shows similar noise level as JPL’s PSD, which is slightly higher than CSR’s PSD but lower than GFZ’s PSD.

3.3 Secular and seasonal gravity variation from global solutions

The primary goal of GRACE mission is to map the temporal variation of Earth’s gravity field, for the purpose to understand the mass transports within the Earth system. Using the same method, we generate a series of monthly gravity solutions up to degree and order 60 from 2003 to 2013. Then we estimate the secular and seasonal variation based on our solution series. Fig. 6(a) shows the estimated secular variation from 2003 to 2013, in terms of equivalent water height (EWH) change. Unlike the geoid map shown in Fig. 3, the EWH map normally contains heavier stripes caused by the amplification of high-frequency noise. Therefore, we applied a 150 km radius Gaussian smoothing (Wahr et al. 1998) to mitigate the error. For comparison, three EWH trend maps from GRACE L2
Figure 4. Global map of both geopotential difference estimates and recovered gravity solution for the month of July 2003. Left-hand panel: geopotential differences (a) estimated from this study (OSU) and predicted from GRACE L2 products of (c) CSR RL05, (e) GFZ RL05a and (g) JPL RL05. Right-hand panel: recovered geoid undulation from (b) this study using geopotential difference estimates (OSU) and from GRACE L2 products of (d) CSR RL05, (f) GFZ RL05a and (g) JPL RL05.
solution series are presented as well in Figs 6(b)–(d), for CSR RL05, GFZ RL05a and JPL RL05, respectively. Clear and consistent secular signals can be observed from Figs 6(a)–(d) for all the solution series, including negative trends in Greenland, Amundsen Sea Embayment, Antarctic Peninsula and Alaska, reflecting the mass loss from the ice sheets and glacier, and positive trends in Hudson Bay, West and East Antarctic, and Scandinavia, mainly due to GIA. We should note that the agreements are mostly over high-latitude and polar regions.

On the other hand, our result seems to have more stripes in certain areas, especially over low-latitude or equatorial regions, such as Southeast Asia with the signal of great Sumatra–Andaman earthquake of 2004 December 26, where our trend map show heavier stripes than CSR. One of the reasons for the discrepancy could be the different reference model used in our processing, especially the high-frequency models, such as ocean tides models, that could regionally impact the aliasing effect if the model is less accurate over certain area. Another reason may be caused by the data coverage of our geopotential difference estimates. Since GRACE is in a near-polar orbit configuration, it is obviously that GRACE data should have more coverage over high-latitude region than middle- and low-latitude region. Therefore, it might be possible to obtain enhanced spatial and/or temporal resolutions over certain region, especially near polar region, using geopotential difference observable derived temporal gravity field solutions.

In general, there is a trade-off between the spatial and temporal resolution for GRACE gravity solutions, which means retrieving a high spatial resolution gravity field needs the compromise of temporal resolution. For example, to get the high degree gravity model (GGM03S, Tapley et al. 2007) up to degree and order 180 (100 km resolution), 4 yr GRACE data are needed. In practice, monthly GRACE solutions are usually truncated to degree 60 (300 km resolution) or so. One exception is that CSR recently generated new monthly solutions (Sakumura 2014) with spherical harmonic coefficients complete to degree 96 (200 km resolution), but it is very challenging to identify high frequency signal beyond degree 60. Dai et al. (2014) used an innovative method to process the degree 96 solutions and successfully retrieved high frequency earthquake signal up to degree 70.
Figure 6. Secular and seasonal gravity variation. Top and middle rows: Trend map (in terms of equivalent water height rate in mm yr\(^{-1}\)) for period from 2003 to 2013 (a) estimated from this study and from GRACE L2 products of (b) CSR RL05, (c) GFZ RL05a and (d) JPL RL05, 150 km Gaussian smoothing applied. Bottom row: Time-series of secular and seasonal mass variation (in Gigaton) for (e) Amazon Basin and (f) Greenland, additional leakage reduction applied.

Figure 7. Residual rms (2003–2013) after removing trend, annual and semi-annual signal from all the monthly solutions, based on the results from this study (OSU) and GRACE L2 products (CSR RL05, GFZ RL05a and JPL RL05).

On the other hand, several attempts have been made to improve the temporal resolution of GRACE. For example, GFZ routinely generates weekly solutions, but only up to degree and order 30. Bruinsma et al. (2010) generate the 10-d solutions up to degree and order 50 with regularization based on traditional dynamic method. Kurtenbach et al. (2009) employed short-arc method (Mayer-Gürr et al. 2007) under the principle of Kalman smoothing to conduct the daily snapshot solution, but the stochastic behaviour of the gravity field has to be considered as \textit{a priori} information. Kang et al. (2008) used traditional dynamic method to generate the so-called ‘quick-look’ solution with a moving-window strategy (with a window step of one day and window width of 15 d), but those solutions are also stabilized using regularization. An incomplete list of GRACE solutions with various temporal resolutions from different research groups can be found at http://icgem.gfz-potsdam.de/ICGEM/TimeSeries.html.
Our improved energy balance approach could also greatly benefit the regional gravity analysis since the geopotential difference data can be easily implemented for gravity recovery with flexible spatial and temporal resolution. In this study, we only discuss the temporal resolution. We choose Greenland as the test region. In last subsection (cf. Fig. 6), we have already estimated the gravity variation over Greenland, but with a temporal resolution of a month and a spatial resolution of degree 60. Here we show that for high-latitude region like Greenland, the temporal resolution can be at least increased by three times without losing the spatial resolution using our improved energy balance approach.

To do that, we collect three months’ geopotential difference data from July 2003 to September 2003, and for each month we split the data into three separate data subsets for every 10 (or 11) d. Then for each 10-d subset, we solve the gravity solution up to degree 60. Neither regularization nor post-processing is applied to these 10-d solutions. In Fig. 8, we present both the data and the solutions, from top to bottom for the month from July 2003 to September 2003. The left three columns of Fig. 8 show the 10-d geopotential difference subset in terms of data coverage map over Greenland, where the colour represents the value with mean field removed; the middle three columns show our 10-d solutions up to degree 60 in terms of geoid undulation with mean field removed; and for comparison the right three columns show the corresponding monthly solutions also up to degree 60 in terms of geoid with mean field removed, from this study, CSR RL05 and GFZ RL05a, respectively. It is interesting to note that although the temporal resolution is shortened to about ten days, the data coverage is still fairly dense over Greenland for most of the subsets. And the resulting 10-d solution, in the middle three columns of Fig. 8 shows explicit geoid variation, surprisingly without introducing more stripes. Correspondingly, the geoid maps over equatorial region from 10-d solutions contains heavier stripes than monthly solutions because of the relative sparse data coverage. Therefore we conclude that, for the region like Greenland, 10-d geopotential difference data are sufficient to recover the time-variable gravity, with a resolution up to degree 60 but without significantly increasing the error. In addition, we can clearly see the geoid fluctuation with each month through three 10-d solutions, such as the geoid decreasing for the month of July. Therefore, these 10-d solutions might also be able to reveal more detailed submonthly temporal mass variations for the Greenland ice-sheet, which is a subject for future studies.

4 CONCLUSIONS AND DISCUSSIONS

Our new, improved energy balance approach is presented in this paper for GRACE data analysis and result interpretation. Compared to previous energy balance studies, the new approach can better preserve gravity information from KBR data and reduce error, especially velocity error, from GPS data, with the help of the novel alignment equation, the reconstruction of reference orbit, and a more accurate formulation of energy equation. As a result, we
obtain more accurate geopotential difference estimates with higher correlation (more than 0.9) with official GRACE RL05 monthly solutions. Using the resulting geopotential difference estimates, we demonstrate the global solution is feasible, and for the first time, produce an independent series of GRACE monthly global solutions based on energy balance principle, with consistent secular and seasonal time-variable gravity signals compared to other solutions based on dynamic method. Furthermore, we show that our improved geopotential difference data can be applied for gravity recovery with flexible temporal resolution, which is conducted over Greenland ice sheets and achieve a higher temporal resolution at 10-d interval, as compared to the official monthly solutions, without the compromise of spatial resolution, nor the need to use regularization or post-processing. It is worth mentioning that our purpose is to only demonstrate that our method mimics the accuracy of L2 products. We are not implying that our global solution based on energy approach is better, that is more accurate and/or higher spatial resolution, than other products.

Our new method would also benefit data analysis of the forthcoming GRACE follow-on mission, especially considering the possibility that the precision of range-rate data can be improved by up to a factor of 20 (Loomis et al. 2012), but the precision of GPS tracking data may not have significant advances. In that case, the weighting of GPS tracking data would need to be further reduced for the traditional conventional dynamic method, which could be more analogous to our method since we have already handle GPS data and range-rate data separately through the alignment equation. Therefore, our energy balance method might have a unique contribution to the processing of more accurate data from next generation satellite gravimetry mission to study mass transports of the Earth.

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REFERENCES


Dimar, P. & van Eck van der Sluijs, A.A., 2004. A technique for Earth’s gravity field modeling on the basis of satellite accelerations, J. Geod., 78, 12–33.


Han, S.-C., 2003. Efficient global gravity determination from satellite-to-satellite tracking (SST), Diss. PhD thesis, Geodetic and Geoinformation Science, Department of Civil and Environmental Engineering and Geodetic Science, The Ohio State University, Columbus, OH, USA.


Pail, R. et al., 2011. First GOCE gravity field models derived by three different approaches, J. Geod., 85(11), 819–843.


Yi, W., 2012. The Earth’s gravity field from GOCE, PhD dissertation, München, Technische Universität München.

APPENDIX: ENERGY EQUATIONS IN INERTIAL AND EARTH-FIXED FRAME

A1. Preliminary

The position vector \( \mathbf{r} \), representing the satellite position relative to Earth centre, can be expanded in any Cartesian coordinate frame \( s \) (called \( s \)-frame) as an ordered triplet of coordinates as \( \mathbf{r} = (r_1, r_2, r_3)^T \), such as inertial frame (i-frame) as \( \mathbf{r}^i \), and Earth-fixed frame (e-frame) as \( \mathbf{r}^e \). The relation between the two coordinate vectors can be described as

\[
\mathbf{r}^e = C_i^e \mathbf{r}^i, \tag{A1}
\]

where \( C_i^e \) is a transformation matrix representing the orientation between the two frames. The time-derivative of \( C_i^e \) can be derived using the angular velocity \( \mathbf{w} \) between the two frames. Here we define \( \mathbf{w}^e_i = (\omega_1, \omega_2, \omega_3)^T \) as the angular velocity vector of the e-frame with respect to the i-frame, with coordinates in the i-frame. The cross-product of the angular velocity vector can be further written as a skew-symmetric matrix

\[
\mathbf{w}^e_i \times = \Omega^e_i = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.
\]

Using above notations, we can derive the time-derivative of the transformation matrix \( C_i^e \) as (Jekeli 2001)

\[
\dot{C}_i^e = -\Omega^e_i C_i^e = \Omega^e_i C_i^e = \mathbf{w}^e_i \times C_i^e = C_i^e \left( \mathbf{w}^e_i \times \mathbf{r}^i \right).
\]

Therefore, taking the time-derivative of (A1) yields

\[
\dot{\mathbf{r}}^e = C_i^e \dot{\mathbf{r}}^i + \mathbf{w}^e_i \times \mathbf{r}^i, \tag{A2}
\]
which represent the transformation of the acceleration vector between two frames. Taking another time-derivative of \((A2)\) yields
\[
\ddot{\mathbf{r}} = C_\varepsilon \dot{\mathbf{r}} + 2C_\varepsilon \mathbf{u}_e \times (\mathbf{w}_e \times \dot{\mathbf{r}}) + C_\varepsilon \left[ \mathbf{w}_e \times (\mathbf{w}_e \times \mathbf{r}) \right],
\]
(A3)
which represents the transformation of the velocity vector between two frames. Here we need to assume \(\mathbf{w}_e = \mathbf{w}_\varepsilon = \dot{\mathbf{w}}_\varepsilon = 0\), which is the first assumption for energy equations.

A2. Newton’s law of motion in both frames

Acceleration vector in \(i\)-frame must obey Newton’s law of motion, which says
\[
\ddot{\mathbf{r}} = \nabla^ The same is for \(\dot{\mathbf{r}}\)-frame as follows:
\[
\dot{\mathbf{r}} = \nabla^ The same is for \(\ddot{\mathbf{r}}\)-frame as follows:
\[
\ddot{\mathbf{r}} = \nabla^ A2. Newton’s law of motion in both frames

Acceleration vector in \(i\)-frame must obey Newton’s law of motion, which says
\[
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