Geoscience Laser Altimeter System (GLAS)

Algorithm Theoretical Basis Document
Version 2.0

PRECISION ORBIT DETERMINATION (POD)

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1.0 INTRODUCTION

1.1 Background

The EOS ICESat mission is scheduled for launch on July 2001. Three major science objectives of this mission are: (1) to measure long-term changes in the volumes (and mass) of the Greenland and Antarctic ice sheets with sufficient accuracy to assess their impact on global sea level, and to measure seasonal and interannual variability of the surface elevation, (2) to make topographic measurements of the Earth's land surface to provide ground control points for topographic maps and digital elevation models, and to detect topographic change, and (3) to measure the vertical structure and magnitude of cloud and aerosol parameters that are important for the radiative balance of the Earth-atmosphere system, and directly measure the height of atmospheric transition layers. The spacecraft features the Geoscience Laser Altimeter System (GLAS), which will measure a laser pulse round-trip time of flight, emitted by the spacecraft and reflected by the ice sheet or land surface. This laser altimeter measurement provides height of the GLAS instrument above the ice sheet. The geocentric height of the ice surface is computed by differencing the altimeter measurement from the satellite height, which is computed from Precision Orbit Determination (POD) using satellite tracking data.

To achieve the science objectives, especially for measuring the ice-sheet topography, the position of the GLAS instrument should be known with an accuracy of 5 cm and 20 cm in radial and horizontal components, respectively. This knowledge will be acquired from data collected by the on-board GPS receiver and from the ground-based satellite laser ranging (SLR) data. GPS data will be the primary tracking data for the ICESat POD, and SLR data will be used for POD validation.
1.2 The POD Problem

The problem of determining an accurate ephemeris for an orbiting satellite involves estimating the position and velocity of the satellite from a sequence of observations which are a function of the satellite position and velocity. This is accomplished by integrating the equations of motion for the satellite from a reference epoch to each observation time to produce predicted observations. The predicted observations are differenced from the true observations to produce observation residuals. The components of the satellite state (satellite position and velocity and the estimated force and measurement model parameters) at the reference epoch are then adjusted to minimize the observation residuals in a least square sense. Thus, to solve the orbit determination problem, one needs the equations of motion describing the forces acting on the satellite, the observation-state relationship describing the relation of the observed parameters to the satellite state, and the least squares estimation algorithm used to obtain the estimate.

An alternative approach available with GPS is a kinematic solution which does not require the description of the dynamics except for possible interpolation between solution points. Kinematic solutions are more sensitive to geometrical factors, such as the direction of the GPS satellites and they require the resolution of phase ambiguities.

This document describes the algorithms for the precise orbit determination (POD) of ICESat/GLAS. Chapter 2 describes the background and objective for ICESat/GLAS mission and POD requirement. Chapter 3 summarizes the dynamic models, and Chapter 4 describes the measurement models for ICESat/GLAS. Chapter
5 describes the least squares estimation algorithm and the problem formulation for multi-satellite orbit determination problem. Chapter 6 summarizes the implementation considerations for ICESat/GLAS, and Chapter 7 summarizes the constraints, limitations, and assumptions of the POD algorithms for ICESat/GLAS.
2.0 OBJECTIVE

The objective of the POD algorithm is to determine an accurate position of the center of mass of the spacecraft carrying the GLAS instrument. This position must be expressed in an appropriate Earth-fixed reference frame, such as the International Earth Rotation Service (IERS) Terrestrial Reference Frame (ITRF), but for some applications the position vector must be given in a non-rotating frame, the IERS Celestial Reference Frame (ICRF). Thus, the POD algorithm will provide a data product that consists of time and the (x,y,z) position (ephemeris) of the spacecraft/GLAS center of mass in both the ITRF and the ICRF. The ephemeris will be provided at an appropriate time interval, e.g., 30 sec and interpolation algorithms will enable determination of the position at any time to an accuracy comparable to the numerical integration accuracy. Furthermore, the transformation matrix between ICRF and ITRF will be provided from the POD, along with interpolation algorithm.
3.0 ALGORITHM DESCRIPTION: Orbit

3.1 ICESat/GLAS Orbit Dynamics Overview

Mathematical models employed in the equations of motion to describe the motion of ICESat/GLAS can be divided into three categories: 1) the gravitational forces acting on ICESat/GLAS consist of Earth’s geopotential, solid earth tides, ocean tides, planetary third-body perturbations, and relativistic accelerations; 2) the non-gravitational forces consist of drag, solar radiation pressure, earth radiation pressure, and thermal radiation acceleration; and 3) the empirical force models could be employed to accommodate unmodeled or mismodeled forces. In this chapter, the dynamic models are described along with the time and reference coordinate systems.

3.2 Equations of Motion, Time and Coordinate Systems

The equations of motion of a near-Earth satellite can be described in an inertial reference frame as follows:

\[ \ddot{\mathbf{r}} = \mathbf{a}_g + \mathbf{a}_{ng} + \mathbf{a}_{emp} \]  (3.2.1)

where \( \ddot{\mathbf{r}} \) is the position vector of the center of mass of the satellite, \( \mathbf{a}_g \) is the sum of the gravitational forces acting on the satellite, \( \mathbf{a}_{ng} \) is the sum of the non-gravitational forces acting on the surfaces of the satellite, and \( \mathbf{a}_{emp} \) is the unmodeled forces which act on the satellite due to either a functionally incorrect description of the various forces acting on
the spacecraft or inaccurate values for the constant parameters which appear in the force model.

3.2.1 Time System

Several time systems are required for the orbit determination problem. From the measurement systems, satellite laser ranging measurements are usually time-tagged in UTC (Coordinated Universal Time) and GPS measurements are time-tagged in GPS System Time (referred to here as GPS-ST). Although both UTC and GPS-ST are based on atomic time standards, UTC is adjusted on the rotation of the Earth. GPS-ST is continuous to avoid complications associated with a discontinuous time scale [Milliken and Zoller, 1978]. Leap seconds are introduced on January 1 or June 1, as required. The relation between GPS-ST and UTC is

\[
GPS-ST = UTC + n
\]

(3.2.2)

where \(n\) is the number of leap seconds since January 6, 1980. For example, the relation between UTC and GPS-ST in mid-July, 1995, was GPS-ST = UTC + 10 sec. The independent variable of the near-Earth satellite equations of motion (Eq. 3.2.1) is typically TDT (Terrestrial Dynamical Time), which is an abstract, uniform time scale implicitly defined by equations of motion. This time scale is related to the TAI (International Atomic Time) by the relation

\[
TDT = TAI + 32.184^s.
\]

(3.2.3)

The planetary ephemerides are usually given in TDB (Barycentric Dynamical Time) scale, which is also an abstract, uniform time scale used as the independent variable for the ephemerides of the Moon, Sun, and planets. The transformation from the TDB
time to the TDT time with sufficient accuracy for most application has been given by Moyer [1981]. For a near-Earth application like ICESat/GLAS, it is unnecessary to distinguish between TDT and TDB. New time systems are under discussion by the International Astronomical Union. This document will be updated with these time systems, as appropriate.

3.2.2 Coordinate System

The inertial reference system adopted for Eq. 3.2.1 for the dynamic model is the ICRF geocentric inertial coordinate system, which is defined by the mean equator and vernal equinox at Julian epoch 2000.0. The Jet Propulsion Laboratory (JPL) DE-405 planetary ephemeris [Standish, 1998], which is based on the ICRF inertial coordinate system, has been adopted for the positions and velocities of the planets with the coordinate transformation from barycentric inertial to geocentric inertial [Thomas et al., 1987].

Tracking station coordinates, atmospheric drag perturbations, and gravitational perturbations are usually expressed in the Earth fixed, geocentric, rotating system, which can be transformed into the ICRF reference frame by considering the precession and nutation of the Earth, the polar motion, and UT1 transformation. The 1976 International Astronomical Union (IAU) precession [Lieske et al., 1977; Lieske, 1979] and the 1980 IAU nutation formula [Wahr, 1981b; Seidelmann, 1982] with the correction derived from VLBI analysis [Herring et al., 1991] has been used as the model of precession and nutation of the Earth. Polar motion and UT1-TAI variation were derived from Lageos (Laser Geodynamics Satellite) laser ranging analysis [Tapley
et al., 1985; Schutz et al., 1988]. Tectonic plate motion for the continental mass on which tracking stations rest has been modeled based on the AM0-2 model [Minster and Jordan, 1978; DeMets et al., 1990; Watkins, 1990]. Yuan [1991] provides more detailed discussion of time and coordinate systems in the satellite orbit determination problem.

3.3 Gravitational Forces

The gravitational forces can be expressed as:

\[
\bar{a}_g = \bar{P}_{geo} + \bar{P}_{st} + \bar{P}_{ot} + \bar{P}_{rd} + \bar{P}_{n} + \bar{P}_{rel}
\]  \hspace{1cm} (3.3.1)

where

- \(\bar{P}_{geo}\) = perturbations due to the geopotential of the Earth
- \(\bar{P}_{st}\) = perturbations due to the solid Earth tides
- \(\bar{P}_{ot}\) = perturbations due to the ocean tides
- \(\bar{P}_{rd}\) = perturbations due to the rotational deformation
- \(\bar{P}_{n}\) = perturbations due to the Sun, Moon and planets
- \(\bar{P}_{rel}\) = perturbations due to the general relativity

3.3.1 Geopotential

The perturbing forces of the satellite due to the gravitational attraction of the Earth can be expressed as the gradient of the potential, \(U\), which satisfies the Laplace equation, \(\nabla^2 U = 0\):
\[ \nabla U = \nabla (U_s + \Delta U_{st} + \Delta U_{ot} + \Delta U_{rd}) = P_{geo} + P_{st} + P_{ot} + P_{rd} \]  
(3.3.2)
where \( U_s \) is the potential due to the solid-body mass distribution, \( \Delta U_{st} \) is the potential change due to solid-body tides, \( \Delta U_{ot} \) is the potential change due to the ocean tides, and \( \Delta U_{rd} \) is the potential change due to the rotational deformations.

The perturbing potential function for the solid-body mass distribution of the Earth, \( U_s \), is generally expressed in terms of a spherical harmonic expansion, referred to as the geopotential, in a body-fixed reference frame as [Kaula, 1966; Heiskanen and Moritz, 1967]:

\[ U_s (r, \phi, \lambda) = \frac{GM_e}{r} + \frac{GM_e}{r} \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left( \frac{a_e}{r} \right)^l \overline{P}_{lm}(\sin \phi) \left[ \overline{C}_{lm} \cos m \lambda + \overline{S}_{lm} \sin m \lambda \right] \]  
(3.3.3)

where

- \( GM_e \) = the gravitational constant of the Earth
- \( a_e \) = the mean equatorial radius of the Earth
- \( \overline{C}_{lm}, \overline{S}_{lm} \) = normalized spherical harmonic coefficients of degree \( l \) and order \( m \)
- \( \overline{P}_{lm}(\sin \phi) \) = the normalized associated Legendre function of degree \( l \) and order \( m \)
- \( r, \phi, \lambda \) = radial distance from the center of mass of the Earth, the geocentric latitude, and the longitude of the satellite

If the origin of the spherical coordinates coincides with the center of mass of the Earth, it follows that \( \overline{C}_{10} = \overline{C}_{11} = \overline{S}_{11} = 0 \).
3.3.2 Solid Earth Tides

Since the Earth is an elastic body to a certain extent, its mass distribution and the shape will be changed under the gravitational attraction of the perturbing bodies, especially the Sun and the Moon. The temporal variation of the free space geopotential induced from the solid Earth tide can be expressed as a change in the external geopotential by the following expression [Wahr, 1981a; Dow, 1988; Casotto, 1989].

\[
\Delta U_{st} = \frac{GM_e}{a^2} \sum_{l=2}^{(3)} \sum_{m=0}^{l} \sum_{k(l,m)} H_k e^{i(\Theta_k+\chi_k)} k_{k0}^0 \left( \frac{a_e}{e} \right)^{l+1} Y^l_m(\phi,\lambda) + k_{k0}^+ \left( \frac{a_e}{e} \right)^{l+3} Y^l_{m+2}(\phi,\lambda)
\]

\[(3.3.4)\]

where

\[Y^l_m(\phi,\lambda) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{lm}(\sin\phi) e^{im\lambda}\]

\[P_{lm}(\sin\phi) = \text{the unnormalized associated Legendre function of degree } l \text{ and order } m\]

\[H_k = \text{the frequency dependent tidal amplitude in meters (provided in Cartwright and Tayler [1971] and Cartwright and Edden [1973])}\]

\[\Theta_k, \chi_k = \text{Doodson argument and phase correction for constituent } k\]

\[(\chi_k = 0, \text{ if } l-m \text{ is even}; \chi_k = -\pi/2, \text{ if } l-m \text{ is odd})\]

\[k_{k0}^0, k_{k0}^+ = \text{Love numbers for constituent } k\]

\[r, \phi, \lambda = \text{geocentric body-fixed coordinates of the satellite}\]

The summation over \(k(l,m)\) means that each different \(l, m\) combination has a unique list of tidal frequencies, \(k\), to sum over.
The tidally induced variations in the Earth’s external potential can be expressed as variations in the spherical harmonic geopotential coefficients [Eanes et al. 1983].

\[
\Delta \bar{C}_{lm} = \frac{(-1)^m}{a_e \sqrt{4\pi(2-\delta_{0m})}} \sum_k k^0 H_k \left\{ \begin{array}{ll}
\cos \Theta_k, & l-m \text{ even} \\
\sin \Theta_k, & l-m \text{ odd}
\end{array} \right.
\]

\[
\Delta \bar{S}_{lm} = \frac{(-1)^m}{a_e \sqrt{4\pi(2-\delta_{0m})}} \sum_k k^0 H_k \left\{ \begin{array}{ll}
\sin \Theta_k, & l-m \text{ even} \\
\cos \Theta_k, & l-m \text{ odd}
\end{array} \right.
\]

where \(\delta_{0m}\) is the Kronecker delta; \(\Delta \bar{C}_{lm}\) and \(\Delta \bar{S}_{lm}\) are the time-varying geopotential coefficients affected by the luni-solar tidal effect.

### 3.3.3 Ocean Tides

The oceanic tidal perturbations due to the attraction of the Sun and the Moon can be expressed as variations in the spherical harmonic geopotential coefficients. The temporal variation of the free space geopotential induced from the ocean tide deformation, \(\Delta U_{ot}\), can be expressed as [Eanes et al., 1983]

\[
\Delta U_{ot} = 4\pi G \rho_w a_e \sum_k \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{n=0}^{1} \frac{1 + k_l^l}{2l+1} \left\{ C_{klm}^\pm \cos(\Theta_k \pm m\lambda) + S_{klm}^\pm \sin(\Theta_k \pm m\lambda) \right\} P_{lm}(\sin \phi)
\]

where \(\rho_w\) is the mean density of sea water, \(k\) is the ocean tide constituent index, \(k_l^l\) is the load Love number of degree \(l\), \(C_{klm}^\pm\) and \(S_{klm}^\pm\) are the unnormalized prograde and retrograde tide coefficients, and \(\Theta_k\) is the Doodson argument for constituent \(k\).
The above variations in the Earth’s external potential due to the ocean tide can be expressed as variations in the spherical harmonic geopotential coefficients as follows [Eanes et al. 1983].

\[ \Delta C_{lm} = F_{lm} \sum_k A_{klm} \]

\[ \Delta S_{lm} = F_{lm} \sum_k B_{klm} \]

(3.3.7)

where \( F_{lm} \), \( A_{klm} \), and \( B_{klm} \) are defined as

\[ F_{lm} = \frac{4\pi a^2 \rho_w}{M_e} \sqrt{\frac{(l+m)!}{(l-m)!(2l+1)(2-\delta_{lm})}} \frac{(1+k_j^l)}{2l+1} \]

(3.3.8)

and

\[
\begin{bmatrix} A_{klm} \\ B_{klm} \end{bmatrix} = \begin{bmatrix} (C_{klm}^+ + C_{klm}^-) \\ (S_{klm}^+ - S_{klm}^-) \end{bmatrix} \cos\Theta_k + \begin{bmatrix} (S_{klm}^+ + S_{klm}^-) \\ (C_{klm}^- - C_{klm}^+) \end{bmatrix} \sin\Theta_k
\]

(3.3.9)

### 3.3.4 Rotational Deformation

Since the Earth is elastic and includes a significant fluid component, changes in the angular velocity vector will produce a variable centrifugal force, which consequently deforms the Earth. This deformation, which is called “rotational deformation”, can be expressed as the change of the centrifugal potential, \( U_c \) [Lambeck, 1980] given by

\[ U_c = \frac{1}{3} \omega^2 r^2 + \Delta U_c \]

(3.3.10)

where

\[ \Delta U_c = \frac{r^2}{6} (\omega_1^2 + \omega_2^2 - 2\omega_3^2) P_{20}(\sin\phi) \]
\[ -\frac{r^2}{3} (\omega_1 \omega_3 \cos \lambda + \omega_2 \omega_3 \sin \lambda) P_{21}(\sin \phi) \]
\[ + \frac{r^2}{12} \left[ (\omega_2^2 - \omega_1^2) \cos 2\lambda - 2 \omega_1 \omega_2 \sin 2\lambda \right] P_{22}(\sin \phi) \]  
(3.3.11)

and \( \omega_1 = \Omega m_1 \), \( \omega_2 = \Omega m_2 \), \( \omega_3 = \Omega (1+m_3) \), and \( \omega^2 = (\omega_1^2 + \omega_2^2 + \omega_3^2) \). \( \Omega \) is the mean angular velocity of the Earth, \( m_i \) are small dimensionless quantities which are related to the polar motion and the Earth rotation parameters by the following expressions:

\[ m_1 = x_p \]
\[ m_2 = -y_p \]  
(3.3.12)
\[ m_3 = \frac{d(UT1-\text{TAI})}{d(\text{TAI})} \]

The first term of Eqs. (3.3.10) is negligible in the variation of the geopotential, and the variation of the free space geopotential outside of the Earth due to the rotational deformation can be written as

\[ \Delta U_{rd} = \left(\frac{a_e}{r}\right)^3 k_2 \Delta U_c(a_e) \]  
(3.3.13)

The above variations in the Earth’s external potential due to the rotational deformation can be expressed as variations in the spherical harmonic geopotential coefficients as follows.

\[ \Delta C_{20} = \frac{a_e^3}{6GM_e} \left[ m_1^2 + m_2^2 - 2(1+m_3)^2 \right] \Omega^2 k_2 = \frac{-a_e^3}{3GM_e} (1+2m_3) \Omega^2 k_2 \]

\[ \Delta C_{21} = \frac{-a_e^3}{3GM_e} m_1 (1+m_3) \Omega^2 k_2 = \frac{-a_e^3}{3GM_e} m_1 \Omega^2 k_2 \]

\[ \Delta S_{21} = \frac{-a_e^3}{3GM_e} m_2 (1+m_3) \Omega^2 k_2 = \frac{-a_e^3}{3GM_e} m_2 \Omega^2 k_2 \]  
(3.3.14)
\[ \Delta C_{22} = \frac{a_e^3}{12GM_e} \left( m_2^2 - m_1^2 \right) \Omega^2 k_2 \approx 0 \]

\[ \Delta S_{22} = \frac{-a_e^3}{6GM_e} \left( m_2 m_1 \right) \Omega^2 k_2 \approx 0 \]

As a consequence of Eqs. (3.3.2), (3.3.3), (3.3.4), (3.3.6), and (3.3.13), the resultant gravitational potential for the Earth can be expressed as

\[ U(r, \phi, \lambda) = \frac{GM_e}{r} + \frac{GM_e}{r} \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left( \frac{a_e^3}{r} \right)^l \bar{P}_{lm}(\sin \phi) \]

\[ \times \left[ (\bar{C}_{lm} + \Delta \bar{C}_{lm}) \cos \lambda + \left( \bar{S}_{lm} + \Delta \bar{S}_{lm} \right) \sin \lambda \right] \quad (3.3.15) \]

where both the solid Earth and oceans contribute to the periodic variations \( \Delta \bar{C}_{lm} \) and \( \Delta \bar{S}_{lm} \).

3.3.5 **N-Body Perturbation**

The gravitational perturbations of the Sun, Moon and other planets can be modeled with sufficient accuracy using point mass approximations. In the geocentric inertial coordinate system, the N-body accelerations can be expressed as:

\[ \bar{P}_n = \sum_i GM_i \left[ \frac{\bar{r}_i}{r_i^3} - \frac{\Delta i}{\Delta i^3} \right] \quad (3.3.16) \]

where

- \( G \) = the universal gravitational constant
- \( M_i \) = mass of the \( i \)-th perturbing body
\( \vec{r}_i \) = position vector of the \( i \)-th perturbing body in geocentric inertial coordinates

\( \vec{\Delta}_i \) = position vector of the \( i \)-th perturbing body with respect to the satellite

The values of \( \vec{r}_i \) can be obtained from the Jet Propulsion Laboratory Development Ephemeris-405 (JPL DE-405) [Standish, 1998].

### 3.3.6 General Relativity

The general relativistic perturbations on the near-Earth satellite can be modeled as [Huang et al., 1990; Ries et al., 1988],

\[
\vec{P}_{rel} = \frac{GM_e}{c^2 r^3} \left\{ \left[ (2\beta+2\gamma) \frac{GM_e}{r} \cdot \vec{\gamma} \cdot \vec{r} \right] \vec{r} + (2+2\gamma) (\vec{r} \cdot \vec{\gamma}) \vec{\gamma} \right\} \\
+ 2 (\vec{\Omega} \times \vec{r}) \\
+ L (1+\gamma) \frac{GM_e}{c^2 r^3} \left[ \frac{3}{r^2} (\vec{r} \times \vec{r}) (\vec{\gamma} \cdot \vec{J}) + (\vec{r} \times \vec{J}) \right]
\]

where

\[
\vec{\Omega} = \frac{(1+\gamma)}{2} (\vec{R}_{ES}) \times \left[ \frac{-GM_s \vec{R}_{ES}}{c^2 R_{ES}^3} \right]
\]

\( c \) = the speed of light in the geocentric frame

\( \vec{r}, \vec{\gamma} \) = the geocentric satellite position and velocity vectors

\( \vec{R}_{ES} \) = the position of the Earth with respect to the Sun

\( GM_e, GM_s \) = the gravitational constants for the Earth and the Sun, respectively

\( \vec{J} \) = the Earth’s angular momentum per unit mass
\[ |\mathbf{J}| = 9.8 \times 10^8 \text{ m}^2/\text{sec} \]

\( L \) = the Lense-Thirring parameter

\( \beta, \gamma \) = the parameterized post-Newtonian (PPN) parameters

The first term of Eqs. (3.3.17) is the Schwarzschild motion [Huang et al., 1990] and describes the main effect on the satellite orbit with the precession of perigee. The second term of Eqs. (3.3.17) is the effect of geodesic (or de Sitter) precession, which results in a precession of the orbit plane [Huang and Ries, 1987]. The last term of Eqs. (3.3.17) is the Lense-Thirring precession, which is due to the angular momentum of the rotating Earth and results in, for example, a 31 mas/yr precession in the node of the Lageos orbit [Ciufolini, 1986].

### 3.4 Nongravitational Forces

The non-gravitational forces acting on the satellite can be expressed as:

\[ \vec{a}_{ng} = \vec{P}_{\text{drag}} + \vec{P}_{\text{solar}} + \vec{P}_{\text{earth}} + \vec{P}_{\text{thermal}} \quad (3.4.1) \]

where

\( \vec{P}_{\text{drag}} \) = perturbations due to the atmospheric drag

\( \vec{P}_{\text{solar}} \) = perturbations due to the solar radiation pressure

\( \vec{P}_{\text{earth}} \) = perturbations due to the Earth radiation pressure

\( \vec{P}_{\text{thermal}} \) = perturbations due to the thermal radiation

Since the surface forces depend on the shape and orientation of the satellite, the models are satellite dependent. In this section, however, general models are described.
3.4.1 Atmospheric Drag

A near-Earth satellite of arbitrary shape moving with some velocity $\mathbf{\bar{v}}$ in an atmosphere of density $\rho$ will experience both lift and drag forces. The lift forces are small compared to the drag forces, which can be modeled as [Schutz and Tapley, 1980]

$$
P_{\text{drag}} = -\frac{1}{2} \rho \left( \frac{C_d A}{m} \right) v_r \mathbf{\bar{v}}_r
$$

(3.4.2)

where

- $\rho$ = the atmospheric density
- $\mathbf{\bar{v}}_r$ = the satellite velocity relative to the atmosphere
- $v_r$ = the magnitude of $\mathbf{\bar{v}}_r$
- $m$ = mass of the satellite
- $C_d$ = the drag coefficient for the satellite
- $A$ = the cross-sectional area of the main body perpendicular to $\mathbf{\bar{v}}_r$

The parameter $\frac{C_d A}{m}$ is sometimes referred to as the ballistic coefficient. Similarly, the drag force on the satellite’s solar panel can be modeled as

$$
P_{\text{panel}} = -\frac{1}{2} \rho \left( \frac{C_{dp} |A_p \cos \gamma|}{m} \right) v_r \mathbf{\bar{v}}_r
$$

(3.4.3)

where

- $C_{dp}$ = the drag coefficient for the solar panel
- $A_p$ = the solar panel’s area
- $\gamma$ = the angle between the solar panel surface normal unit vector, $\mathbf{\hat{n}}$, and satellite velocity vector, $\mathbf{\bar{v}}_r$ (i.e. $\cos \gamma = \mathbf{\hat{n}} \cdot \frac{\mathbf{\bar{v}}_r}{|\mathbf{\bar{v}}_r|}$)
- $|A_p \cos \gamma|$ = the effective solar panel area perpendicular to $\mathbf{\bar{v}}_r$
There are a number of empirical density models used for computing the atmospheric density. These include the Jacchia 71 [Jacchia, 1971], Jacchia 77 [Jacchia, 1977], and the Drag Temperature Model (DTM) [Barlier et al., 1977]. The density computed by using any of these models could be in error anywhere from 10% to over 200% depending on solar activity [Shum et al., 1986]. To account for the deviations in the computed values of density from the true density, the computed values of density, $\rho_c$, can be modified by using empirical once-per-revolution density correction parameters [Elyasberg et al., 1972; Shum et al., 1986] such as

$$\rho = \rho_c [1 + C_1 \cos(M+\omega) + C_2 \sin(M+\omega)]$$  \hspace{1cm} (3.4.4)

where

- $C_1, C_2$ = the once-per-revolution density correction coefficients
- $M$ = mean anomaly of the satellite
- $\omega$ = argument of perigee of the satellite

### 3.4.2 Solar Radiation Pressure

The Sun emits a nearly constant amount of photons per unit of time. At a mean distance of 1 A.U. from the Sun, this radiation pressure is characterized as a momentum flux having an average value of $4.56 \times 10^{-6} \text{ N/m}^2$. The direct solar radiation pressure from the Sun on a satellite is modeled as [Tapley and Ries, 1987]

$$\bar{P}_{\text{solar}} = -P (1 + \eta) \frac{A}{m} v \hat{u}$$  \hspace{1cm} (3.4.5)

where

- $P$ = the momentum flux due to the Sun
\[ \eta \] = reflectivity coefficient of the satellite
\[ A \] = the cross-sectional area of the satellite normal to the Sun
\[ m \] = mass of the satellite
\[ \nu \] = the eclipse factor (\( \nu = 0 \) if the satellite is in full shadow, \( \nu = 1 \) if the satellite is in full Sun, and \( 0 < \nu < 1 \) if the satellite is in partial shadow)
\[ \hat{u} \] = the unit vector pointing from the satellite to the Sun

Similarly, the solar radiation pressure perturbation on the satellite’s solar panel can be modeled as

\[
P_{\text{panels}} = -P \nu \frac{|A_p \cos \gamma|}{m} \left( \hat{u} + \eta_n \hat{n} \right)
\] (3.4.6)

where
\[ A_p \] = the solar panel area
\[ \hat{n} \] = the surface normal unit vector of the solar panel
\[ \gamma \] = the angle between the solar panel surface normal unit vector, \( \hat{n} \), and satellite-Sun unit vector, \( \hat{u} \) (i.e. \( \cos \gamma = \hat{u} \cdot \hat{n} \))
\[ |A_p \cos \gamma| \] = the effective solar panel area perpendicular to \( \hat{u} \)

The reflectivity coefficient, \( \eta \), represents the averaged effect over the whole satellite rather than the actual surface reflectivity. Conical or cylindrical shadow models for the Earth and the lunar shadow are used to determine the eclipse factor, \( \nu \). Since there are discontinuities in the solar radiation perturbation across the shadow boundary, numerical integration errors occur for satellites which are in the shadowing region. The
modified back differences (MBD) method [Anderle, 1973] can be implemented to account for these errors [Lundberg, 1985; Feulner, 1990].

3.4.3 Earth Radiation Pressure

Not only the direct solar radiation pressure, but also the radiation pressure imparted by the energy flux of the Earth should be modeled for the precise orbit determination of any near-Earth satellite. The Earth radiation pressure model can be summarized as follows [Knocke and Ries, 1987; Knocke, 1989].

\[
\vec{P}_{\text{earth}} = (1 + \eta_e) A^\prime \frac{A_c}{m c} \sum_{j=1}^{N} \left[ (\tau a E_s \cos \theta_s + e M_B) \hat{r} \right]_j
\]

(3.4.7)

where

\( \eta_e \) = satellite reflectivity for the Earth radiation pressure
\( A^\prime \) = the projected, attenuated area of a surface element of the Earth
\( A_c \) = the cross sectional area of the satellite
\( m \) = the mass of the satellite
\( c \) = the speed of light
\( \tau \) = 0 if the center of the element \( j \) is in darkness
1 if the center of the element \( j \) is in daylight
\( a, e \) = albedo and emissivity of the element \( j \)
\( E_s \) = the solar momentum flux density at 1 A.U.
\( \theta_s \) = the solar zenith angle
\( M_B \) = the exitance of the Earth
\( \hat{r} \) = the unit vector from the center of the element \( j \) to the satellite
N = the total number of segments

This model is based on McCarthy and Martin [1977].

The nominal albedo and emissivity models can be represented as

\[ a = a_0 + a_1 P_{10}(\sin \phi) + a_2 P_{20}(\sin \phi) \]  (3.4.8)

\[ e = e_0 + e_1 P_{10}(\sin \phi) + e_2 P_{20}(\sin \phi) \]  (3.4.9)

where

\[ a_1 = c_0 + c_1 \cos \omega(t-t_0) + c_2 \sin \omega(t-t_0) \]  (3.4.10)

\[ e_1 = k_0 + k_1 \cos \omega(t-t_0) + k_2 \sin \omega(t-t_0) \]  (3.4.11)

and

\[ P_{10}, P_{20} = \text{the first and second degree Legendre polynomial} \]

\( \phi \) = the latitude of the center of the element on the Earth’s surface

\( \omega \) = frequency of the periodic terms (period = 365.25 days)

\( t-t_0 \) = time from the epoch of the periodic term

This model, based on analyses of Earth radiation budgets by Stephens et al. [1981], characterizes both the latitudinal variation in Earth radiation and the seasonally dependent latitudinal asymmetry.

### 3.4.4 Thermal Radiation Perturbation

Since the temperatures of the satellite’s surface are not uniform due to the internal and external heat fluxes, there exists a force due to a net thermal radiation imbalance. This perturbation depends on the shape, the thermal property, the pattern of
thermal dumping, the orbit characteristics, and the thermal environment of the satellite as a whole. A detailed discussion of the thermal radiation perturbation models for Lageos can be found in Ries [1989]. It is observed for GPS satellites that there are unexplained forces in the body-fixed +Y or -Y direction, that is along solar panel rotation axis, which causes unmodeled accelerations [Fliegel et al., 1992]. This acceleration is referred to as the “Y-bias”. Possible causes of the Y-bias are solar panel axis misalignment, solar sensor misalignment, and the heat generated in the GPS satellite body, which is radiated preferentially from louvers on the +Y side. Since this Y-bias perturbation is not predictable, it can be modeled as

$$
F_{bias} = \alpha \cdot \mathbf{u}_Y
$$

(3.4.12)

where $\mathbf{u}_Y$ is a unit vector in the Y-direction, and the scale factor, $\alpha$, is estimated for each GPS satellite.

### 3.5 Empirical Forces

To account for the unmodeled forces which act on the satellite, some empirical forces are modeled in Eqs. (3.3.1). Those include the empirical tangential perturbation and the once-per-orbital-revolution acceleration in the radial, transverse, and normal directions.
3.5.1 Empirical Tangential Perturbation

Unmodeled forces in the tangential direction, either along the inertial velocity or along the body-fixed velocity, may be estimated by using empirical models during the orbit determination process. This tangential perturbation can be modeled empirically as

\[ \mathbf{P}_{\text{tangent}} = C_t \hat{u}_t \]  

(3.5.1)

where

- \( C_t \) = empirical tangential parameter
- \( \hat{u}_t \) = the unit vector in the tangential direction (along inertial velocity or body-fixed velocity)

Such forces are included when it is believed there are non-conservative forces of unknown origin acting in the tangential direction. A set of piecewise constants, \( C_t \), can be estimated to account for this unmodeled tangential perturbations.

3.5.2 Once-per Revolution RTN Perturbation

Unmodeled perturbations in the radial, transverse, and normal directions can be modeled as

\[ \mathbf{P}_{\text{rm}} = \begin{bmatrix} P_r \\ P_t \\ P_n \end{bmatrix} = \begin{bmatrix} C_r \cos u + S_r \sin u \\ C_t \cos u + S_t \sin u \\ C_n \cos u + S_n \sin u \end{bmatrix} \]  

(3.5.2)

where

- \( P_r \) = once-per rev. radial perturbation
- \( P_t \) = once-per rev. transverse perturbation
These empirical perturbations, which are computed in the radial, transverse, and normal components, are transformed into the geocentric inertial components.
4.0 ALGORITHM DESCRIPTION: Measurements

4.1 ICESat/GLAS Measurements Overview

The GPS measurements will be the primary measurement type for the ICESat/GLAS POD, while the laser range measurement will serve as a secondary source of verification and evaluation of the GPS-based ICESat/GLAS POD product. In this chapter, the mathematical models of the GPS measurements and the laser range measurement are discussed.

4.2 GPS Measurement Model

The GPS measurements are ranges which are computed from measured time or phase differences between received signals and receiver generated signals. Since these ranges are biased by satellite and receiver clock errors, they are called as pseudoranges. In this section, code pseudorange (PR) measurements, phase pseudorange measurements (PPR), double-differenced high-low phase pseudorange measurements (DDHL) which involve one ground station, two GPS satellites, and one low Earth orbiting satellite, are discussed. Hofmann-Wellenhof et al. [1992] and Remondi [1984] are referred for more discussion of GPS measurement models.

4.2.1 Code Pseudorange Measurement

The PR measurement, \( \rho_{PR} \), can be modeled as follows,

\[
\rho_{PR}^C = \rho - c \cdot \delta t_t + c \cdot \delta t_r + \delta \rho_{trop} + \delta \rho_{iono} + \delta \rho_{rel} \quad (4.2.1)
\]
where \( r \) is the slant range between the GPS satellite and the receiver, \( c \) is the speed of light, \( \delta t_i \) is the GPS satellite's clock error, \( \delta r \) is the receiver's clock error, \( \delta \rho_{trop} \) is the tropospheric delay, \( \delta \rho_{iono} \) is the ionospheric delay, and \( \delta \rho_{rel} \) is the correction for relativistic effect.

### 4.2.2 Phase Pseudorange Measurement

The carrier phase measurement between a GPS satellite and a ground station can be modeled as follows,

\[
\phi^c_i(t_R) = \phi^j(t_T) - \phi^i(t_R) + N_{ij}(t_0) \tag{4.2.2a}
\]

where \( t_R \) is the received time of the \( i \)-th ground receiver, \( t_T \) is the transmitted time of the \( j \)-th satellite’s phase being received by the \( i \)-th receiver at \( t_R \), \( \phi^c_i(t_R) \) is the computed phase difference between the \( j \)-th GPS satellite and \( i \)-th ground receiver at \( t_R \), \( \phi^j(t_T) \) is the phase of \( j \)-th GPS satellite received by \( i \)-th receiver, \( \phi^i(t_R) \) is the phase of \( i \)-th ground receiver at \( t_R \), \( t_0 \) is the initial epoch of the \( i \)-th receiver, and \( N_{ij}(t_0) \) is the integer bias which is unknown. Similarly, the carrier phase measurement between a GPS satellite and a low satellite can be modeled as follows,

\[
\phi^c_u(t_{R_u}) = \phi^j(t_{T_u}) - \phi^u(t_{R_u}) + N_{uj}(t_0) \tag{4.2.2b}
\]

where \( t_{R_u} \) is the received time of the on-board receiver of the user satellite, \( t_{T_u} \) is the transmitted time of the \( j \)-th satellite’s phase being received by the user satellite at \( t_{R_u} \), \( \phi^c_u(t_{R_u}) \) is the computed phase difference between \( j \)-th GPS satellite and the user satellite at \( t_{R_u} \), \( \phi^j(t_{T_u}) \) is the phase of \( j \)-th GPS satellite received by the user satellite,
\( \phi_i(t_{R_i}) \) is the phase of the user satellite at \( t_{R_i} \), \( t_0 \) is the initial epoch of the user satellite, and \( N_u(t_0) \) is the unknown integer bias.

The signal transmitted time of the \( j \)-th GPS satellite can be related to the signal received time by

\[
\begin{align*}
    t_{T_i}^j &= t_{R_i} - (\rho_i^j(t_{R_i})/c) - \delta t_{\phi_i}^j \\
    t_{T_u}^j &= t_{R_u} - (\rho_u^j(t_{R_u})/c) - \delta t_{\phi_u}^j
\end{align*}
\]

where \( \rho_i^j \) is the geometric line of sight range between \( j \)-th GPS satellite and \( i \)-th ground receiver, \( \rho_u^j \) is the slant range between \( j \)-th GPS satellite and the on-board receiver of the user satellite, \( \delta t_{\phi_i}^j \) is the sum of ionospheric delay, tropospheric delay, and relativistic effect on the signal traveling from \( j \)-th GPS satellite to \( i \)-th ground receiver, \( \delta t_{\phi_u}^j \) is the sum of ionospheric delay, tropospheric delay, and relativistic effect on the signal traveling from \( j \)-th satellite to the on-board receiver of the user satellite. Since the time tag, \( t_i \) or \( t_u \), of the measurement is in the receiver time scale which has some clock error, the true received times are

\[
\begin{align*}
    t_{R_i} &= t_i - \delta t_c \\
    t_{R_u} &= t_u - \delta t_{c_u}
\end{align*}
\]

where \( \delta t_c \) is the clock error of the \( i \)-th ground receiver at \( t_R \) and \( \delta t_{c_u} \) is the clock error of the on-board receiver of the user satellite at \( t_{R_u} \). Since the satellite oscillators and the receiver oscillators are highly stable clocks, the \((1\sigma)\) change of the frequency over the specified period, \( \Delta f \), is on the order of \( 10^{-12} \). With such high stability, the linear approximation of \( \phi(t + \delta t) = \phi(t) + f \cdot \delta t \) can be used for \( \delta t \) which is usually less
By substituting Eqs.(4.2.3a) and (4.2.4a) into Eqs.(4.2.2a), and neglecting higher order terms, Eqs.(4.2.2a) becomes

\[ \phi^c_j(t_R) = \phi^i(t_i) - f^j \cdot [\delta t_c + \rho^j(t_R)/c + \delta t_{\phi^j}] \]

\[ - \phi_i(t_i) + f_i \delta t_c + N_i^j(t_0) \] (4.2.5a)

Similarly, the phase measurement between \( j \)-th GPS satellite and the user satellite can be modeled as follows,

\[ \phi^u_j(t_R) = \phi^u(t_u) - f^j \cdot [\delta t_c + \rho^j(t_R)/c + \delta t_{\phi^j}] \]

\[ - \phi_u(t_u) + f_u \delta t_c + N_u^j(t_0) \] (4.2.5b)

By multiplying a negative nominal wave length, \(-\lambda = -c/f_0\), where \( f_0 \) is the nominal value for both the transmitted frequency of the GPS signal and the receiver mixing frequency, Eqs.(4.2.5a) becomes the phase pseudorange measurement,

\[ PPR^c_j = \frac{f^j}{f_0} \rho^j(t_R) + \frac{f^j}{f_0} \delta \rho_{\phi^j} + \frac{f^j}{f_0} c \delta t_c - \frac{f_i}{f_0} c \delta t_c \]

\[ - \frac{c}{f_0} \cdot \left[ \phi^i(t_i) - \phi_i(t_i) \right] + C_i^j \] (4.2.6)

where \( \delta \rho_{\phi^j} = c \delta t_{\phi^j} \) and \( C_i^j = -\left( \frac{c}{f_0} \right) \cdot N_i^j \).

The first term of second line of Eqs. (4.2.6) can be expanded using the following relations.

\[ \phi^i(t_i) - \phi_i(t_i) = \phi^i(t_0) - \phi_i(t_0) + \int_{t_0}^{t_i} (f^j - f_i) dt \] (4.2.7)
However, \( \phi^j(t_0) - \phi_i(t_0) = f^j \delta t_c^j(t_0) - f_i \delta t_c(t_0) \), which is the time difference between the satellite and the receiver clocks at the first data epoch, \( t_0 \). And \( \int_{t_0}^{t_i} (f^j - f_i) \, dt \) is the total number of cycles the two oscillators have drifted apart over the interval from \( t_0 \) to \( t_i \). According to Remondi [1984], this is equivalent to the statement that the two clocks have drifted apart, timewise, by amount \( \left[ \delta t_c^j(t_i) - \delta t_c(t_i) \right] - \left[ \delta t_c^j(t_0) - \delta t_c(t_0) \right] \). Thus,

\[
\phi^j(t_i) - \phi_i(t_i) = f^j \cdot \delta t_c^j - f_i \cdot \delta t_c_i
\]  

(4.2.8)

After substituting Eqs. (4.2.8), Eqs. (4.2.6) becomes,

\[
PPR^{c \, j} = \frac{f^j}{f_0} \rho^j((R_i) + \frac{f^j}{f_0} \delta \rho_{\phi^j} - \frac{f^j}{f_0} c \ \delta t_c^j + \frac{f^j}{f_0} c \ \delta t_{c_i} + C_{ij}
\]  

(4.2.9a)

Similarly, the phase pseudorange between \( j \)-th satellite and a user satellite can be written as,

\[
PPR^{c \, u} = \frac{f^j}{f_0} \rho^j((R_u) + \frac{f^j}{f_0} \delta \rho_{\phi^j} - \frac{f^j}{f_0} c \ \delta t_c^j + \frac{f^j}{f_0} c \ \delta t_{c_u} + C_{uj}
\]  

(4.2.9b)

Since the GPS satellites have highly stable oscillators which have 10^{-11} or 10^{-12} clock drift rate, the frequencies of those clocks usually stay close to the nominal frequency, \( f_0 \). If the frequencies are expressed as \( f^j = f_0 + \Delta f^j \), where \( \Delta f \) is clock frequency offset from the nominal value, Eqs.(4.2.9a) and (4.2.9b) become as follows after ignoring negligible terms.

\[
PPR^{c \, j} = \rho^j((R_i) + \delta \rho_{\phi^j} - c \ \delta t_c^j + c \ \delta t_{c_i} + C_{ij}
\]  

(4.2.10a)

\[
PPR^{c \, u} = \rho^j((R_u) + \delta \rho_{\phi^j} - c \ \delta t_c^j + c \ \delta t_{c_u} + C_{uj}
\]  

(4.2.10b)

Note that \( \rho^j((R_i) \) and \( \rho^j((R_u) \) could be expanded as
\[ \rho_i^j(t_{R_i}) = \rho_i^j(t_i) - \dot{\rho}_i^j \delta t_{c_i} \]  \hfill (4.2.11a) \\
\[ \rho_u^j(t_{R_u}) = \rho_u^j(t_u) - \dot{\rho}_u^j \delta t_{c_u} \]  \hfill (4.2.11b) 

Thus, Eqs. (4.2.10a) and (4.2.10b) become

\[ PPR_i^j = \rho_i^j(t_i) + \delta \rho_{\phi_i}^j - c \delta t_{c_i}^j + c \delta t_{c_i} - \dot{\rho}_i^j \delta t_{c_i} + C_i^j \]  \hfill (4.2.12a) \\
\[ PPR_u^j = \rho_u^j(t_u) + \delta \rho_{\phi_u}^j - c \delta t_{c_u}^j + c \delta t_{c_u} - \dot{\rho}_u^j \delta t_{c_u} + C_u^j \]  \hfill (4.2.12b) 

Eqs. (4.2.12a) is the phase pseudorange measurement between a ground receiver and a GPS satellite, and Eqs. (4.2.12b) is the phase pseudorange measurement between a GPS satellite and a user satellite. Note that the clock errors would be estimated for each observation epoch.

### 4.2.3 Double-Differenced High-Low Phase Pseudorange Measurement

By subtracting Eqs.(4.2.2b) from Eqs.(4.2.2a), a single-differenced high-low phase measurement can be formed as follows,

\[ SDHLP_{i/u}^c = \phi_i^c(t_{R_i}) - \phi_u^c(t_{R_u}) \]  \hfill (4.2.13) 

If another single-differenced high-low phase measurement can be obtained between \( i \)-th ground receiver, \( k \)-th GPS satellite, and the user satellite, a double-differenced high-low phase measurement can be formed by subtracting those two single-differenced high-low phase measurements.

\[ DDHLP_{i/u}^{c/k} = - f^i \left[ \delta t_{c_i} + \rho_i^j(t_{R_i})/c + \delta t_{\phi_i}^j \right] \\
+ f^u \left[ \delta t_{c_u} + \rho_u^j(t_{R_u})/c + \delta t_{\phi_u}^j \right] \]
where $N_{i ju}^k = N_i^j(t_0) - N_u^j(t_0) - N_i^k(t_0) + N_u^k(t_0)$. In Eqs.(4.2.14), all the phase terms associated with ground station and user satellite receiver are canceled out.

By multiplying a negative nominal wave length, $-\lambda = -c/f_0$, Eqs.(4.2.14) becomes the double-differenced high-low phase pseudorange measurement,

$$DDHL_{iu}^{j k} = \left( \frac{f^j}{f_0} \right) (\rho_i^j(t_R) - \rho_u^j(t_R)) - \left( \frac{f^k}{f_0} \right) (\rho_i^k(t_R) - \rho_u^k(t_R))$$

$$- \left( \frac{c}{f_0} \right) \cdot (\phi_i^j(t_i) - \phi_u^j(t_i) - \phi^j(t_u) + \phi^k(t_u))$$

$$+ c \cdot \left( \frac{f^j - f^k}{f_0} \right) (\delta t_{ci} - \delta t_{cu})$$

$$+ \left( \frac{f^j}{f_0} \right) \cdot (\delta \rho_{\phi_i}^j - \delta \rho_{\phi_u}^j) - \left( \frac{f^k}{f_0} \right) \cdot (\delta \rho_{\phi_i}^k - \delta \rho_{\phi_u}^k)$$

$$+ C_{iu}^{j k}$$  (4.2.15)

where $\delta \rho_{\phi} = -c \cdot \delta t_{\phi}$, and $C_{iu}^{j k} = -\lambda \cdot N_{iu}^{j k}$. Note that Eqs.(4.2.15) contains two different time tags, $t_i$ and $t_u$. If the ground station receiver clock and the on-board receiver clock are synchronized, then the second line can be canceled out. However, the ICESat/GLAS on-board receiver clock runs freely without having any
synchronization with ground receiver clock, and samples the data at every second. Since the difference in the time tags, $\Delta t_{iu}$, is less than or equal to 0.5 seconds, linear approximation can be made for the second line of Eqs.(4.2.15). By substituting $t_u = t_i + \Delta t_{iu}$, Eqs.(4.2.15) becomes,

$$DDHL_{i u}^{j k} = \frac{f_j}{f_0} (\rho_{i j}(t_R) - \rho_{u j}(t_{R_u})) - \frac{f_k}{f_0} (\rho_{i k}(t_R) - \rho_{u k}(t_{R_u}))$$

$$+ c \cdot \left( \frac{f_j - f_k}{f_0} \right) \cdot \Delta t_{iu}$$

$$+ c \cdot \left( \frac{f_j - f_k}{f_0} \right) \left( \delta t_{c_i} - \delta t_{c_u} \right)$$

$$+ \left( \frac{f_j}{f_0} \right) (\delta \rho_{\phi j} - \delta \rho_{\phi u}) + \left( \frac{f_k}{f_0} \right) (\delta \rho_{\phi k} - \delta \rho_{\phi u})$$

$$+ C_{i u}^{j k} \quad (4.2.16)$$

Since the GPS satellites have highly stable oscillators which have $10^{-11}$ or $10^{-12}$ clock drift rate, the frequencies of those clocks usually stay close to the nominal frequency, $f_0$. If the frequencies are expressed as $f^j = f_0 + \Delta f^j$ and $f^k = f_0 + \Delta f^k$, Eqs.(4.2.16) becomes

$$DDHL_{i u}^{j k} = \rho_{i j}(t_R) - \rho_{u j}(t_{R_u}) - \rho_{i k}(t_R) + \rho_{u k}(t_{R_u})$$

$$\left( \frac{\Delta f^j}{f_0} \right) (\rho_{i j}(t_R) - \rho_{u j}(t_{R_u})) - \left( \frac{\Delta f^k}{f_0} \right) (\rho_{i k}(t_R) - \rho_{u k}(t_{R_u}))$$

$$+ c \cdot \left( \frac{\Delta f^j - \Delta f^k}{f_0} \right) \cdot \Delta t_{iu}$$

$$+ c \cdot \left( \frac{\Delta f^j - \Delta f^k}{f_0} \right) \left( \delta t_{c_i} - \delta t_{c_u} \right)$$
For the ICESat/GLAS-GPS case, the single differenced range can be 600 km to 9000 km. If we assume $10^{-11}$ clock drift rate for GPS satellite clocks, the second line contributes an effect, which is at the sub-millimeter level to the double differenced range measurement. This effect is less than the noise level, and as a consequence, the contribution from the second line can be ignored. The third line also is negligible for the ICESat/GLAS-GPS case, since $\Delta t_{uu}$ is less than 0.5 second. Since the performance specification of the time-tag errors of the flight and ground receivers for ICESat/GLAS mission is required to be less than 0.5 microsecond with respect to the GPS time, the fourth line also is negligible. The sixth line is totally negligible, because even for the propagation delay of 100 m, the contribution from this line is less than $10^{-9}$ meters. The first line in Eqs. (4.2.17) can be expanded by the linear approximation after substituting Eqs. (4.2.4a) and (4.2.4b), to obtain:

$$DDHL_{iu}^{jk} = \rho_i^j(t_i) - \rho_u^j(t_u) - \rho_i^k(t_i) + \rho_u^k(t_u)$$

$$- \left[ \dot{\rho}_i^j(t_i) - \dot{\rho}_u^k(t_u) \right] \cdot \delta t_{ei} + \left[ \dot{\rho}_u^j(t_u) - \dot{\rho}_u^k(t_u) \right] \cdot \delta t_{eu}$$

$$+ \delta \rho_{\phi_i}^j - \delta \rho_{\phi_u}^j - \delta \rho_{\phi_i}^k + \delta \rho_{\phi_u}^k$$

$$+ C_{ij}^{j,k} \quad (4.2.18)$$
This equation can be implemented for the double-differenced high-low phase pseudorange measurement. The second line does not need to be computed if the ground stations and the ICESat/GLAS on-board receiver’s time-tags are corrected in the preprocessing stage by using independent clock information from the pseudo-range measurement. If such clock information is not available, then the receiver clock errors, $\delta t_c$ and $\delta t_{cu}$, can be modeled as linear functions,

$$\delta t_{ci} = a_i + b_i (t_i - t_{i0}) \quad (4.2.19a)$$

$$\delta t_{cu} = a_u + b_u (t_u - t_{u0}) \quad (4.2.19b)$$

where $(a_i, b_i)$ and $(a_u, b_u)$ are pairs of clock bias and clock drift for $i$-th ground station receiver clock and the user satellite clock, respectively, and $t_{i0}$ and $t_{u0}$ are the reference time for clock parameters for $i$-th ground station receiver clock and the user satellite clock.

The third line of Eqs. (4.2.18) includes the propagation delay and the relativistic effects for the high-low phase converted measurement. These effects are discussed in more detail in the following sections. It should be noted that Selective Availability (SA) is not considered in the derivation of Eqs. (4.2.18). For the real ICESat/GLAS-GPS data processing, however, interpolation is used to aid in synchronizing clocks to assume SA cancellation.
4.2.4 Corrections

4.2.4.1 Propagation Delay

When a radio wave is traveling through the atmosphere of the Earth, it experiences a delay due to the propagation refraction. Atmospheric scientists usually divide the atmosphere into four layers. These are the troposphere, the stratosphere, the mesosphere, and the thermosphere. The troposphere, the lowest layer of the Earth’s atmosphere, contains 99% of the atmosphere’s water vapor and 90% of the air. The troposphere is composed of dry and wet components. The dry path delay is caused by the atmosphere gas content in the troposphere while the wet path delay is caused by the water vapor content. Since the tropospheric path delay of a radio wave is frequency independent, this path delay cannot be removed by using different frequencies. The tropospheric delays caused by the dry portion, which accounts for 80% or more of the delay, can be modeled with an accuracy of two to five percent for L-band frequencies [Atmospheric & Kalaghan, 1974]. Although the contribution from the wet component is relatively small, it is more difficult to model because surface measurements of water vapor cannot provide accurate estimates of the spatial variation of its water vapor content. There are several approaches to model the wet component of the tropospheric path delay. One approach is to use one of the empirical atmospheric models based on the measurement of meteorological parameters at the Earth’s surface or the altitude profile with radiondes. The other approach is to measure the water vapor content directly with the measuring devices such as water vapor radiometer (WVR). List of references for these approaches can be found in Tralli et al. [1988]. Chao’s model
[Chao, 1974], modified Hopfield model [Goad and Goodman, 1974; Remondi, 1984], or MTT model [Herring, 1992] can be implemented for the tropospheric correction.

The ionosphere is a region of the Earth’s upper atmosphere, approximately 100 km to 1000 km above the Earth’s surface, where electrons and ions are present in quantities sufficient to affect the propagation of radio waves. The path delay will be proportional to the number of electrons along the slant path between the satellite and the receiver, and the electron density distribution varies with altitude, time of day time of year, and solar sunspot cycle. The ionospheric path delay depends on the frequency of the radio signal. This ionospheric effects on L1 GPS measurement will vary from about 0.15 m to 50 m [Clynch and Coco, 1986]. Some of this delay can be eliminated by ionospheric modeling [for example, Finn and Matthewman, 1989]. However, more accurate corrections can be made by using the dual frequency measurements. The correction method for the dual frequency GPS measurements can be found in Ho [1990]. Hofmann-Wellenhof et. al. [1992] provides more detailed description of propagation delay for GPS measurements.

4.2.4.2 Relativistic Effect

The relativistic effects on GPS measurements can be summarized as follows. Due to the difference in the gravitational potential between the satellite and the user, the satellite clock tends to run faster than the ground station’s [Spilker, 1978; Gibson, 1983]. These effects can be divided into a constant drift part and a periodic part. The constant drift part can be removed by setting the GPS clock frequency low
before launch to account for that constant drift. The periodic relativistic effects can be modeled for a high-low measurement as

$$
\Delta \rho_{srel} = \frac{2}{c} \left( \vec{r}_l \cdot \vec{v}_l - \vec{r}_h \cdot \vec{v}_h \right) \tag{4.2.20}
$$

where

- $\Delta \rho_{srel}$ = correction for the special relativity
- $c$ = speed of light
- $\vec{r}_l, \vec{v}_l$ = the position and velocity of the low satellite or tracking stations
- $\vec{r}_h, \vec{v}_h$ = the position and velocity of the high satellite

The coordinate speed of light is reduced when light passes near a massive body causing a time delay, which can be modeled as [Holdridge, 1967]

$$
\Delta \rho_{grel} = (1 + \gamma) \frac{GM_e}{c^2} \ln \left( \frac{r_{tr} + r_{rec} + \rho}{r_{tr} + r_{rec} - \rho} \right) \tag{4.2.21}
$$

where

- $\Delta \rho_{grel}$ = correction for the general relativity
- $\gamma$ = the parameterized post-Newtonian (PPN) parameter ($\gamma = 1$ for general relativity)
- $GM_e$ = gravitational constant for the Earth
- $\rho$ = the relativistically uncorrected range between the transmitter and the receiver
- $r_{tr}$ = the geocentric radial distance of the transmitter
- $r_{rec}$ = the geocentric radial distance of the receiver
4.2.4.3 Phase Center Offset

The geometric offset between the transmitter and receiver phase centers and the effective ICESat/GLAS reference point can be modeled depending on the satellite orientation (attitude) and spacecraft geometry. The specific ICESat/GLAS antenna location depends on the final design, however, the location of the antenna phase center with respect to the spacecraft center of mass will be required. This position vector will be essentially constant in spacecraft fixed axes, but this correction is necessary since the equations of motion refer to the spacecraft center of mass.

4.2.4.4 Ground Station Related Effects

In computing the double-differenced phase-converted high-low range measurement, it is necessary to consider the effects of the displacement of the ground station location caused by the crustal motions. Among these effects, tidal effects and tectonic plate motion effects are the major ones to consider.

Tidal effects can be divided into three parts as

\[ \Delta_{\text{tide}} = \Delta_{\text{diide}} + \Delta_{\text{ocean}} + \Delta_{\text{rotate}} \]  (4.2.22)

where

- \( \Delta_{\text{tide}} \) = the total displacement due to the tidal effects
- \( \Delta_{\text{diide}} \) = the displacement due to the solid Earth tide
- \( \Delta_{\text{ocean}} \) = the displacement due to the ocean loading
- \( \Delta_{\text{rotate}} \) = the displacement due to the rotational deformation
The approach of the IERS Standards [McCarthy, 1996] can be implemented for the solid Earth tide correction. Ocean loading effects are due to the elastic response of the Earth’s crust to ocean tides. The displacement due to the rotational deformation is the displacement of the ground station by the elastic response of the Earth’s crust to shifts in the spin axis orientation [Goad, 1980]. Detailed models for the effects of solid Earth tide, the ocean loading, and the rotational deformation, can be found in Yuan [1991].

The effect of the tectonic plate motion, which is based on the relative plate motion model AM0-2 of Minster and Jordan [1978], can be modeled as

$$\Delta_{tect} = (\bar{\omega}_{p} \times \bar{R}_{s_0}) (t_i - t_0)$$  \hspace{1cm} (4.2.23)

where

- $\Delta_{tect}$ = the displacement due to the tectonic motion
- $\bar{\omega}_{p}$ = the angular velocity of the tectonic plate
- $\bar{R}_{s_0}$ = the Earth-fixed coordinates of the station at $t_i$
- $t_0$ = a reference epoch

### 4.2.5 Measurement Model Partial Derivatives

The partial derivatives of Eqs. (4.2.18) with respect to various model parameters are derived in this section. The considered parameters include the ground station positions, GPS satellite’s positions, user satellite’s positions, clock parameters, ambiguity parameters, and tropospheric refraction parameters.
The partial derivatives of Eqs. (4.2.18) with respect to the \( i \)-th ground station positions, \((x_{1i}, x_{2i}, x_{3i})\), are:

\[
\frac{\partial \text{DDHL}^{c,j,k}_{i,u}}{\partial x_{mi}} = \frac{(x_{mi} - x_{mj})}{\rho_i^j} - \frac{(x_{mi} - x_{mk})}{\rho_i^k} , \quad \text{for } m=1,2,3 \tag{4.2.24}
\]

where \( \rho_i^j \) is the range between \( i \)-th ground station receiver and \( j \)-th transmitter, and \( \rho_i^k \) is the range between \( i \)-th ground station receiver and \( k \)-th transmitter such that:

\[
\rho_i^j = \sqrt{(x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2 + (x_{3i} - x_{3j})^2} \tag{4.2.25}
\]

\[
\rho_i^k = \sqrt{(x_{1i} - x_{1k})^2 + (x_{2i} - x_{2k})^2 + (x_{3i} - x_{3k})^2} \tag{4.2.26}
\]

and \((x_{1j}, x_{2j}, x_{3j})\) and \((x_{1k}, x_{2k}, x_{3k})\) are the \( j \)-th and \( k \)-th transmitter Cartesian positions, respectively.

The partial derivatives of Eqs. (4.2.18) with respect to the \( j \)-th and \( k \)-th transmitter positions are:

\[
\frac{\partial \text{DDHL}^{c,j,k}_{i,u}}{\partial x_{mj}} = -\frac{(x_{mi} - x_{mj})}{\rho_i^j} + \frac{(x_{mu} - x_{mj})}{\rho_u^j} , \quad \text{for } m=1,2,3 \tag{4.2.27}
\]

\[
\frac{\partial \text{DDHL}^{c,j,k}_{i,u}}{\partial x_{mk}} = \frac{(x_{mi} - x_{mk})}{\rho_i^k} - \frac{(x_{mu} - x_{mk})}{\rho_u^k} , \quad \text{for } m=1,2,3 \tag{4.2.28}
\]

where \( \rho_u^j \) is the range between \( j \)-th transmitter and the user satellite, and \( \rho_u^k \) is the range between \( k \)-th transmitter and the user satellite such that:

\[
\rho_u^j = \sqrt{(x_{1u} - x_{1j})^2 + (x_{2u} - x_{2j})^2 + (x_{3u} - x_{3j})^2} \tag{4.2.29}
\]

\[
\rho_u^k = \sqrt{(x_{1u} - x_{1k})^2 + (x_{2u} - x_{2k})^2 + (x_{3u} - x_{3k})^2} \tag{4.2.30}
\]
and \((x_{1u}, x_{2u}, x_{3u})\) is the user satellite’s Cartesian positions.

The partial derivatives of Eqs. (4.2.18) with respect to the user satellite positions are

\[
\frac{\partial DDHL c_{ij}^k}{\partial x_{mu}} = -\frac{(x_{mu} - x_{m^j})}{\rho_u^j} + \frac{(x_{mu} - x_{m^k})}{\rho_u^k}, \quad \text{for } m=1,2,3 \tag{4.2.31}
\]

The partial derivatives of Eqs. (4.2.18) with respect to the clock parameters of Eqs. (4.2.19a) and (4.2.19b) are

\[
\frac{\partial DDHL c_{ij}^k}{\partial a_i} = -(\dot{\rho}_i^j - \dot{\rho}_i^k) \tag{4.2.32}
\]

\[
\frac{\partial DDHL c_{ij}^k}{\partial b_i} = -(\dot{\rho}_i^j - \dot{\rho}_i^k) \cdot (t_i - t_{i0}) \tag{4.2.33}
\]

and

\[
\frac{\partial DDHL c_{ij}^k}{\partial a_u} = (\dot{\rho}_u^j - \dot{\rho}_u^k) \tag{4.2.34}
\]

\[
\frac{\partial DDHL c_{ij}^k}{\partial b_u} = (\dot{\rho}_u^j - \dot{\rho}_u^k) \cdot (t_u - t_{u0}) \tag{4.2.35}
\]

The partial derivative of Eqs. (4.2.18) for the double-differenced ambiguity parameter, \(C_{ij}^k\), is

\[
\frac{\partial DDHL c_{ij}^k}{\partial C_{ij}^k} = 1 \tag{4.2.36}
\]
When Chao’s model is used, the partial derivative of Eqs. (4.2.18) with respect to the \(i\)-th ground station’s zenith delay parameter, \(Z_i\), is

\[
\frac{\partial DDHL_{cij}^{k}}{\partial Z_i} = \left( \frac{1}{\sin E_{ij} + \frac{0.00143}{\tan E_{ij} + 0.0445}} + \frac{1}{\sin E_{ik} + \frac{0.00035}{\tan E_{ik} + 0.017}} \right) 
- \left( \frac{1}{\sin E_{ij} + \frac{0.00143}{\tan E_{ij} + 0.0445}} + \frac{1}{\sin E_{ik} + \frac{0.00035}{\tan E_{ik} + 0.017}} \right)
\]

(4.2.37)

where \(E_{ij}\) and \(E_{ik}\) are the elevation angles of the \(j\)-th and \(k\)-th transmitter from \(i\)-th ground station, respectively.

4.3 SLR Measurement Model

4.3.1 Range Model and Corrections

The laser range measurement is the travel time required from the reference point of the laser ranging instrument to the satellite retroreflector and back to the optical receiver at the tracking station. The one-way range from the reference point of the ranging instrument to the retroreflector of the satellite, \(r^o\), can be expressed in terms of the round trip light time, \(\Delta \tau\) as

\[
r^o = \frac{1}{2} c \Delta \tau + \epsilon
\]

(4.3.1)

where

\[
c = \text{the speed of light} \]

\[ \epsilon = \text{measurement error}. \]

The computed one-way signal path between the reference point on the satellite and the ground station, \( \rho^c \), can be expressed as

\[ \rho^c = |\bar{r} - \bar{r}_s| + \Delta \rho_{trop} + \Delta \rho_{grel} + \Delta \rho_{c.m.} \]  \hspace{1cm} (4.3.2)

where
\[ \bar{r} = \text{the satellite position in geocentric coordinates} \]
\[ \bar{r}_s = \text{the position of the tracking station in geocentric coordinates} \]
\[ \Delta \rho_{trop} = \text{correction for tropospheric delay} \]
\[ \Delta \rho_{grel} = \text{correction for the general relativity} \]
\[ \Delta \rho_{c.m.} = \text{correction for the offset of the satellite's center-of-mass and the laser retroreflector} \]

The tropospheric refraction correction is computed using the model of Marini and Murray [1973]. The correction for the general relativity for SLR measurement is the same as for GPS measurement, which is expressed in Eqs. (4.2.21). The effects of the displacement of the ground station location caused by the crustal motions should be considered. These crustal motions include tidal effects and tectonic plate motion effects, which are described in Eqs. (4.2.22) and (4.2.23), respectively.

### 4.3.2 Measurement Model Partial Derivatives

The partial derivatives of Eqs. (4.3.2) with respect to various model parameters are derived in this section. The considered parameters include the ground station positions, satellite’s positions.
The partial derivatives of Eqs. (4.3.2) with respect to the ground station positions, \((r_{s1}, r_{s2}, r_{s3})\), are

\[
\frac{\partial \rho^c}{\partial r_{si}} = \frac{(r_{si} - r_i)}{\rho}, \quad \text{for } i=1,2,3 \quad (4.3.3)
\]

where \((r_1, r_2, r_3)\) are the satellite's positions, and \(\rho\) is the range between the ground station and the satellite such that

\[
\rho = \sqrt{(r_1 - r_{s1})^2 + (r_2 - r_{s2})^2 + (r_3 - r_{s3})^2} \quad (4.3.4)
\]

The partial derivatives of Eqs. (4.3.2) with respect to the satellite's positions, \((r_1, r_2, r_3)\), are

\[
\frac{\partial \rho^c}{\partial r_i} = \frac{(r_i - r_{si})}{\rho}, \quad \text{for } i=1,2,3 \quad (4.3.5)
\]
5.0 ALGORITHM DESCRIPTION: Estimation

A least square batch filter [Tapley, 1973] is one approach for the estimation procedure. Since multi-satellite orbit determination problems require extensive usage of computer memory for computation, it is essential to consider the computational efficiency in problem formulation. This section describes the estimation procedures for ICESat/GLAS POD, including the problem formulation for multi-satellite orbit determination.

5.1 Least Squares Estimation

The equations of motion for the satellite can be expressed as

\[ \dot{X}(t) = F(X, t), \quad X(t_0) = X_0 \quad (5.1.1) \]

where \( X \) is the \( n \)-dimensional state vector, \( F \) is a non-linear \( n \)-dimensional vector function of the state, and \( X_0 \) is the value of the state at the initial time \( t_0 \), which is usually not known perfectly. The observations consist of discrete measurements of quantities which are a function of the state. Thus the observation-state relationship can be written as

\[ Y_i = G(X_i, t_i) + \epsilon_i \quad i = 1, \ldots, l \quad (5.1.2) \]

where \( Y_i \) is a \( p \) vector of the observations made at time \( t_i \), \( (X_i, t_i) \) is a non-linear vector function relating the state to the observations, and \( \epsilon_i \) is the measurement noise.

If a reference trajectory is available and if \( X \), the true trajectory, and \( X^* \), the reference trajectory, remain sufficiently close throughout the time interval of interest,
the trajectory for the actual motion can be expanded in a Taylor series about the reference trajectory to obtain a set of differential equations with time dependent coefficients. Using a similar procedure to expand the nonlinear observation-state relation, a linear relation between the observation deviation and the state deviation can be obtained. Then, the nonlinear orbit determination problem can be replaced by a linear orbit determination problem in which the deviation from the reference trajectory is to be determined.

Let

$$x(t) = X(t) - X^*(t) \quad y(t) = Y(t) - Y^*(t)$$

(5.1.3)

where $X^*(t)$ is a specified reference trajectory and $Y^*(t)$ is the value of the observation calculated by using $X^*(t)$. Then, substituting Eq. (5.1.3) into Eqs. (5.1.1) and (5.1.2), expanding in a Taylor’s series, and neglecting terms of order higher than the first leads to the relations

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

(5.1.4)

$$y_i = \tilde{H}\dot{x}_i + \epsilon_i \quad i = 1, \ldots, l$$

where

$$A(t) = \frac{\partial F(X^*, t)}{\partial X} \quad \tilde{H} = \frac{\partial G(X^*, t)}{\partial X}$$

(5.1.5)

The general solution to Eq. (5.1.4) can be expressed as

$$x(t) = \Phi(t, t_0)x_0$$

(5.1.6)
where the state transition matrix $\Phi(t, t_0)$ satisfies the differential equation:

$$\Phi(t, t_0) = A(t)\Phi(t_0), \quad \Phi(t_0) = I \quad (5.1.7)$$

where $I$ is the $n \times n$ identity matrix.

Using Eq. (5.1.5), the second of Eq. (5.1.3) may be written in terms of the state at $t_0$ as

$$y_i = \tilde{H}_i \Phi(t_i, t_0) x_0 + \epsilon_i, \quad i = 1, \ldots, l \quad (5.1.8)$$

Using the solution for the linearized state equation (Eq. (5.1.6)), Eq. (5.1.8) may be rewritten as

$$y = Hx_0 + \epsilon \quad (5.1.9)$$

where

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix}, \quad H = \begin{bmatrix} \tilde{H}_1 \Phi(t_1, t_0) \\ \vdots \\ \tilde{H}_l \Phi(t_l, t_0) \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_l \end{bmatrix} \quad (5.1.10)$$

where $y$ and $\epsilon$ are $m$ vectors ($m = l \times p$) and $H$ is an $m \times n$ matrix. Equation (5.1.9) is a system of $m$ equations in $n$ unknowns. In practical orbit determination problems, there are more observations than estimated parameters ($m > n$), which means that Eq. (5.1.9) is overdetermined. It is usually assumed that the observation error vector, $\epsilon$, satisfies the a priori statistics, $E[\epsilon] = 0$ and $E[\epsilon \epsilon^T] = W^{-1}$. By scaling each term in Eq. (5.1.9) by $W^{1/2}$, the condition

$$W^{1/2}[\epsilon \epsilon^T]W^{T/2} = W^{1/2}W^{-1}W^{T/2} = I \quad (5.1.11)$$
is obtained.

An approach to obtain the best estimate of \( \hat{x} \), given the linear observation-state relations (Eq. (5.1.9)) is described in the following discussions. The method obtains the solution by applying successive orthogonal transformations to the linear equations given in Eq. (5.1.9). Consider the quadratic performance index

\[
J = \frac{1}{2} \| W^{1/2} (Hx - y) \|^2 = \frac{1}{2} (Hx - y)^T W (Hx - y) \tag{5.1.12}
\]

The solution to the weighted least-squares estimation problem (which is equivalent to the minimum variance and the maximum likelihood estimation problem, under certain restrictions) is obtained by finding the value \( \hat{x} \) which minimizes Eq. (5.1.12). To achieve the minimum value of Eq. (5.1.12) let \( Q \) be an \( m \times m \) orthogonal matrix. Hence, it follows that Eq. (5.1.12) can be expressed as

\[
J = \frac{1}{2} \| Q W^{1/2} (Hx - y) \|^2 \tag{5.1.13}
\]

Now, if \( Q \) is selected such that

\[
Q W^{1/2} H = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad Q W^{1/2} y = \begin{bmatrix} b \\ e \end{bmatrix} \tag{5.1.14}
\]

where \( R \) is \( n \times n \) upper-triangular, \( 0 \) is an \( (m-n) \times n \) null matrix, \( b \) is \( n \times 1 \) vector, and \( e \) is an \( (m-n) \times 1 \) vector. Equation (5.1.13) can be written then as

\[
J(x) = \frac{1}{2} \| Rx - b \|^2 + \frac{1}{2} \| e \|^2 \tag{5.1.15}
\]

The value of \( x \) which minimizes Eq. (5.1.12) is obtained by the solution

\[
R \hat{x} = b \tag{5.1.16}
\]

and the minimum value of the performance index becomes
\[ J(\hat{x}) = \frac{1}{2} |e|^2 = \frac{1}{2} \| y - H\hat{x} \|^2 \]  

(5.1.17)

That is, \( e \) provides an estimate of the residual error vector.

The procedures is direct and for mechanization requires only that a convenient computational procedure for computing \( QW^{1/2}H \) and \( QW^{1/2}y \) be available. The two most frequently applied methods are the Givens method, based on a sequence of orthogonal rotations, and the Householder method, based on a series of orthogonal reflections [Lawson and Hanson, 1974].

In addition to the expression for computing the estimate, the statistical properties of the error in the estimate, \( R \), are required. If the error in the estimate, \( \eta \), is defined as

\[ \eta = \hat{x} - x \]  

(5.1.18)

it follows that

\[ E[\eta] = E[\hat{x} - x] = E[R^{-1}b - x] \]  

(5.1.19)

Since

\[ QW^{1/2}y = QW^{1/2}Hx + QW^{1/2}e \]

leads to

\[ b = Rx + \tilde{e} \]  

(5.1.20)

it follows that

\[ E[\eta] = E[R^{-1}(Rx + \tilde{e}) - x] = E[R^{-1}\tilde{e}] \]  

(5.1.21)

As noted in Eq. (5.1.11), if the observation error, \( \varepsilon \), is unbiased, \( \tilde{e} = QW^{1/2}\varepsilon \) will be unbiased and
\[ E[\eta] = 0 \]  

Hence, \( \hat{x} \) will be an unbiased estimate of \( x \). Similarly, the covariance matrix for the error in \( x \) can be expressed as

\[ P = E[\eta\eta^T] \]

\[ = E[R^{-1}\tilde{e}\tilde{e}^T R^{-T}] = R^{-1}E[\tilde{e}\tilde{e}^T]R^{-T} \]  

(5.1.23)

If the observation error, \( e \), has a statistical covariance defined as \( E[ee^T] = W^{-1} \), the estimation error covariance matrix is given by

\[ E[\tilde{e}\tilde{e}^T] = W^{1/2}E[ee^T]W^{T/2} = W^{1/2}W^{-1}W^{T/2} = I. \]  

Consequently, relation (5.1.23) leads to

\[ P = R^{-1}R^{-T} \]  

(5.1.24)

It follows then that the estimate of the state and the associated error covariance matrix are given by the expressions

\[ \hat{x} = R^{-1}b \]  

(5.1.25)

\[ P = R^{-1}R^{-T} \]  

(5.1.26)

### 5.2 Problem Formulation for Multi-Satellite Orbit Determination

Proper categorization of the parameters will help to clarify the problem formulation. Parameters can be divided into two groups as dynamic parameters and kinematic parameters. Dynamic parameters are those that need to be mapped into other states by using the state transition matrix which is usually computed by numerical
integration, while kinematic parameters are the ones which will be constant throughout
the computation. Dynamic parameters can be grouped again into two parts as the local
dynamic parameters and global dynamic parameters. Local dynamic parameters are
dynamic parameters which depend on each satellite only. Global dynamic parameters
are dynamic parameters which influence every satellite.

Following the categorization described above, the estimation state vector is

\[
X \equiv \begin{bmatrix}
X_{KP} \\
X_{SS} \\
X_{LDP} \\
X_{GDP}
\end{bmatrix}
\]  \hspace{1cm} (5.2.1)

where

- \(X_{KP}\) = the kinematic parameters \((n_{kp})\)
- \(X_{SS}\) = the satellite states \((n_{ss})\)
- \(X_{LDP}\) = the local dynamic parameters \((n_{ldp})\)
- \(X_{GDP}\) = the global dynamic parameters \((n_{gdp})\)

and \(X_{SS}\) consists of satellites’ positions and velocities, i.e. \(X_{SS} \equiv [X_{POS}, X_{VEL}]^T\). For
\(ns\)-satellites, where \(ns\) is the total number of satellites which will be estimated, \(X_{SS}\) becomes

\[
X_{SS} = \begin{bmatrix}
\vec{r}_1 \\
\vec{r}_2 \\
\vdots \\
\vec{r}_{ns} \\
\vec{v}_1 \\
\vec{v}_2 \\
\vdots \\
\vec{v}_{ns}
\end{bmatrix}
\]
where $\bar{r}_i$ and $\bar{v}_i$ are the 3×1 position and velocity vectors of the $i$-th satellite, respectively.

The differential equations of state, Eq. (5.1.1), becomes

$$
\dot{X}(t) = F(X,t) = \begin{bmatrix} 0 & X_{SS} \end{bmatrix}, \quad X(t_0) = X_0
$$

(5.2.2)

where

$$
X_{SS} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{ns} \\ \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_{ns} \end{bmatrix}
$$

(5.2.3)

and $\bar{f}_i = \bar{a}_{gi} + \bar{a}_{ng}$ for $i$-th satellite. Eq. (5.2.2) represent a system of $n$ nonlinear first order differential equations which includes $n_{ss} = 6 \times ns$ of Eq. (5.2.3). After the linearization process described in section 5.1, Eq. (5.2.2) becomes Eqs. (5.1.6) and (5.1.7).

Since Eq. (5.1.7) represents $n^2$ coupled first order ordinary differential equations, the dimension of the integration vector becomes $n_{ss} + n^2$. However, $A(t)$ matrix is a sparse matrix, because of the nature of the parameters. And $A(t)$ matrix becomes even more sparse, since each satellite’s state is independent of the others, i.e. $(\bar{r}_j, \bar{v}_j)$ is independent of $(\bar{r}_j, \bar{v}_j)$ for $i \neq j$. Using the partitioning of Eq. (5.2.1), $A(t)$ becomes
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & A_{32} & A_{33} & A_{34} & A_{35} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(5.2.4)

where

\[
A_{32} = \begin{bmatrix}
\frac{\partial \vec{f}_1}{\partial \vec{r}_1} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial \vec{r}_{\text{ns}}} \\
0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial \vec{r}_{\text{ns}}} & \cdots & \cdots
\end{bmatrix}
\]

\[
A_{33} = \begin{bmatrix}
\frac{\partial \vec{f}_1}{\partial \vec{v}_1} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial \vec{v}_{\text{ns}}} \\
0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial \vec{v}_{\text{ns}}} & \cdots & \cdots
\end{bmatrix}
\]

\[
A_{34} = \begin{bmatrix}
\frac{\partial \vec{f}_1}{\partial X_{LDP}} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial X_{LDP_{\text{ns}}}} \\
0 & \cdots & \frac{\partial \vec{f}_{\text{ns}}}{\partial X_{LDP_{\text{ns}}}} & \cdots & \cdots
\end{bmatrix}
\]

\[
A_{35} = \begin{bmatrix}
\frac{\partial \vec{f}_1}{\partial X_{GDP}} \\
\vdots \\
\frac{\partial \vec{f}_{\text{ns}}}{\partial X_{GDP_{\text{ns}}}}
\end{bmatrix}
\]

Note that \(A_{32}, A_{33}, \text{ and } A_{34}\) are all block diagonal matrix, and \(A_{33}\) would be zero if the perturbations do not depend on satellites’ velocity. Atmospheric drag is one example of perturbations which depend on the satellite’s velocity.

If \(\Phi = [\phi_{ij}]\), for \(i, j = 1, \cdots, 5\), Eq. (5.1.7) becomes
\[
\Phi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} & \phi_{35} \\
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (5.2.5)

where \( B_{1j} = A_{32}\phi_{2j} + A_{33}\phi_{3j} + A_{34}\phi_{4j} + A_{35}\phi_{5j} \) for \( j = 1, \ldots, 5 \).

Integrating the first row and last two rows of Eq. (5.2.4) with the initial conditions, \( \Phi(t_0,t_0) = I \) yields the results that \( \phi_{11}=\phi_{44}=\phi_{55}=I \) and \( \phi_{12}=\phi_{13}=\phi_{14}=\phi_{15}=\phi_{41}=\phi_{42}=\phi_{43}=\phi_{45}=\phi_{51}=\phi_{52}=\phi_{53}=\phi_{54}=0 \). After substituting these results to \( B_{1j}, j = 1, \ldots, 5 \), we have

\[
B_{11} = A_{32}\phi_{21} + A_{33}\phi_{31}
\]

\[
B_{12} = A_{32}\phi_{22} + A_{33}\phi_{32}
\]

\[
B_{13} = A_{32}\phi_{23} + A_{33}\phi_{33}
\] (5.2.6)

\[
B_{14} = A_{32}\phi_{24} + A_{33}\phi_{34} + A_{34}
\]

\[
B_{15} = A_{32}\phi_{25} + A_{33}\phi_{35} + A_{35}
\]

From Eq. (5.2.5) and Eq. (5.2.6), we have

\[
\phi_{21} = \phi_{31}
\] (5.2.7a)

\[
\phi_{22} = \phi_{32}
\] (5.2.8a)
\[ \phi_{23} = \phi_{33} \quad (5.2.9a) \]

\[ \phi_{24} = \phi_{34} \quad (5.2.10a) \]

\[ \phi_{25} = \phi_{35} \quad (5.2.11a) \]

\[ \phi_{31} = A_{32}\phi_{21} + A_{33}\phi_{31} \quad (5.2.7b) \]

\[ \phi_{32} = A_{32}\phi_{22} + A_{33}\phi_{32} \quad (5.2.8b) \]

\[ \phi_{33} = A_{32}\phi_{23} + A_{33}\phi_{33} \quad (5.2.9b) \]

\[ \phi_{34} = A_{32}\phi_{24} + A_{33}\phi_{34} + A_{34} \quad (5.2.10b) \]

\[ \phi_{35} = A_{32}\phi_{25} + A_{33}\phi_{35} + A_{35} \quad (5.2.11b) \]

From Eqs. (5.2.7a) and (5.2.7b)

\[ \ddot{\phi}_{21} - A_{33}\dot{\phi}_{21} - A_{32}\phi_{21} = 0, \quad \phi_{21}(0) = 0, \quad \dot{\phi}_{21}(0) = 0 \quad (5.2.12) \]

If we define the partials of accelerations w.r.t. each group of parameters for the \(i\)-th satellite as follows,

\[ \frac{\partial \ddot{r}_i}{\partial r_i} \equiv DADR_i \quad (5.2.13a) \]

\[ \frac{\partial \ddot{r}_i}{\partial v_i} \equiv DADV_i \quad (5.2.13b) \]
\[
\frac{\partial f_i}{\partial X_{LDP_i}} \equiv DLDP_i
\]  
(5.2.13c)

\[
\frac{\partial f_i}{\partial X_{GDP_i}} \equiv DGDP_i
\]  
(5.2.13d)

and \(\phi_2\) is partitioned as \(\phi_2 = [\phi_{21}, \cdots, \phi_{21_n}]^T\) by \(3 \times n_k p\) submatrix, \(\phi_2\), then, Eq. (5.2.12) become

\[
\ddot{\phi}_{21} - DADV_i \dot{\phi}_{21} - DADR_i \phi_{21} = 0, \quad i = 1, \cdots, n_s
\]  
(5.2.14)

After applying the initial conditions, \(\phi_{21}(0) = 0\) and \(\dot{\phi}_{21}(0) = 0\), to Eq. (5.2.14), we have \(\phi_{21} = 0\). And from Eq. (5.2.7a) \(\phi_{31} = 0\). From Eqs. (5.2.8a) and (5.2.8b), and Eqs. (5.2.9a) and (5.2.9b), we have similar results as follows.

\[
\ddot{\phi}_{22} - DADV_i \dot{\phi}_{22} - DADR_i \phi_{22} = 0, \quad i = 1, \cdots, n_s
\]  
(5.2.15)

\[
\ddot{\phi}_{23} - DADV_i \dot{\phi}_{23} - DADR_i \phi_{23} = 0, \quad i = 1, \cdots, n_s
\]  
(5.2.16)

with the initial conditions \(\phi_{22}(0) = I\), \(\dot{\phi}_{22}(0) = 0\), \(\phi_{23}(0) = 0\), and \(\dot{\phi}_{23}(0) = I\) for \(i = 1, \cdots, n_s\).

From Eqs. (5.2.10a) and (5.2.10b), we have

\[
\ddot{\phi}_{24} - A_{33} \dot{\phi}_{24} - A_{32} \phi_{24} = A_{34}, \quad \phi_{24}(0) = 0 \quad \dot{\phi}_{24}(0) = 0
\]  
(5.2.17)
If $\phi_{24}$ is partitioned as $\phi_{24} = [\phi_{241}, \cdots, \phi_{24n}]^T$ with $3 \times n_{ldp}$ submatrix, where $n_{ldp}$ is the $i$-th satellite’s number of local dynamic parameters, then it can be shown that all the off-block diagonal terms become zero and the above equation becomes,

$$\ddot{\phi}_{24i} - DADV_i \dot{\phi}_{24i} - DADR_i \phi_{24i} = DLDP_i, \quad i = 1, \cdots, ns \quad (5.2.18)$$

with the initial conditions $\phi_{24i}(0) = 0$ and $\dot{\phi}_{24i}(0) = 0$ for $i = 1, \cdots, ns$.

From Eqs. (5.2.11a) and (5.2.11b), we have similar results for $\phi_{25}$.

$$\ddot{\phi}_{25i} - DADV_i \dot{\phi}_{25i} - DADR_i \phi_{25i} = DGDP_i, \quad i = 1, \cdots, ns \quad (5.2.19)$$

with the initial conditions $\phi_{25i}(0) = 0$ and $\dot{\phi}_{25i}(0) = 0$ for $i = 1, \cdots, ns$.

Combining all these results, we have the state transition matrix for multi-satellite problem as follows:

$$\Phi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} \\
\dot{\phi}_{21} & \dot{\phi}_{22} & \dot{\phi}_{23} & \dot{\phi}_{24} & \dot{\phi}_{25} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (5.2.20)$$

where $\phi_{21} = \phi_{21} = 0$ and

$$\phi_{22} = \begin{bmatrix}
\phi_{221} & 0 \\
\vdots & \ddots \\
0 & \phi_{22n}
\end{bmatrix}, \quad \phi_{23} = \begin{bmatrix}
\phi_{231} & 0 \\
\vdots & \ddots \\
0 & \phi_{23n}
\end{bmatrix}$$
By defining $\phi_r$ and $\phi_v$ for $i$-th satellite as follows,

$$
\phi_{24} = \begin{bmatrix}
\phi_{24i} & 0 \\
\vdots & \ddots \\
0 & \phi_{24ns}
\end{bmatrix} \quad \phi_{25} = \begin{bmatrix}
\phi_{25i} \\
\vdots \\
\phi_{25ns}
\end{bmatrix}
$$

we can compute $\phi_{v_i} = \begin{bmatrix}
\dot{\phi}_{22i} \\
\dot{\phi}_{23i} \\
\dot{\phi}_{24i} \\
\dot{\phi}_{25i}
\end{bmatrix}$ by substituting Eqs. (5.2.15)-(5.2.16) and Eqs. (5.2.18)-(5.2.19).

$$
\ddot{\phi}_{v_i} = \begin{bmatrix}
DADV_i \phi_{22i} + DADR_i \phi_{22i} \\
DADV_i \phi_{23i} + DADR_i \phi_{23i} \\
DADV_i \phi_{24i} + DADR_i \phi_{24i} + DLDP_i \\
DADV_i \phi_{25i} + DADR_i \phi_{25i} + DGDP_i
\end{bmatrix}
$$

After rearranging this equation, we get

$$
\dot{\phi}_{v_i} = DADV_i \phi_{v_i} + DADR_i \phi_r + [0_{3x3} 0_{3x3} DLDP_i DGDP_i]
$$

Eq. (5.2.23) represents $3 \times (6+n_{ldp}+n_{gdp})$ first order differential equations for the $i$-th satellite. Therefore, the total number of equations for $ns$ satellites becomes $\sum_{i=1}^{ns} 3 \times (6+n_{ldp}+n_{gdp})$. 
Since multi-satellite orbit determination problem includes different types of satellites in terms of their perturbations and integration step size, a class of satellite is defined as a group of satellites which will use the same size of geopotential perturbation and the same integration order and step size. For $l$-classes of satellites, the integration vector, $X_{INT}$, is defined as
where \( ns_i \) is the number of satellites for \( i \)-th class, \( \vec{r}_{ij} \) and \( \vec{v}_{ij} \) are the position and velocity of the \( j \)-th satellite of \( i \)-th class, respectively. \( \phi_{T_{ij}} \) is the state transition matrix
for the \( j \)-th satellite’s positions of \( i \)-th class and \( \phi_{V_{ij}} \) is the state transition matrix for the \( j \)-th satellite’s velocities of \( i \)-th class.

\[
\begin{bmatrix}
\ddots \\
\bar{V}_{11} \\
\vdots \\
\bar{V}_{1\text{ns}_1}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\phi_{V_{11}} \\
\vdots \\
\phi_{V_{1\text{ns}_1}}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\bar{f}_1 \\
\vdots \\
\bar{f}_{1\text{ns}_1}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\phi_{\bar{V}_{11}} \\
\vdots \\
\phi_{\bar{V}_{1\text{ns}_1}}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\bar{V}_1 \\
\vdots \\
\bar{V}_{\text{ns}_1}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\phi_{V_1} \\
\vdots \\
\phi_{V_{\text{ns}_1}}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\bar{f}_1 \\
\vdots \\
\bar{f}_{\text{ns}_1}
\end{bmatrix}
\begin{bmatrix}
\phi_{V_{j1}} \\
\vdots \\
\phi_{V_{j\text{ns}_1}}
\end{bmatrix}
\begin{bmatrix}
\ddots \\
\bar{f}_1 \\
\vdots \\
\bar{f}_{\text{ns}_1}
\end{bmatrix}
\begin{bmatrix}
\phi_{\bar{V}_{j1}} \\
\vdots \\
\phi_{\bar{V}_{j\text{ns}_1}}
\end{bmatrix}
\]

\( \chi_{INT} = \) (5.2.25)
Eq. (5.2.25) is numerically integrated using a procedure such as the Krogh-Shampine-Gordon fixed-step fixed-order formulation for second-order differential equations [Lundberg, 1981] for each class of satellites. For the ICESat/GLAS-GPS case, two classes of satellites need to be defined. One is for the high satellites, i.e. GPS, and the other is for the low satellite, i.e. ICESat/GLAS.

5.3 Output

Although a large number of parameters are available from the estimation process as given by Eqs. (5.2.24), the primary data product required for the generation of other products is the ephemeris of the ICESat/GLAS spacecraft center of mass. This ephemeris will be generated at a specified interval, e.g., 30 sec and will include the following:

- $t$ in GPS time
- $(X,Y,Z)$ of the spacecraft center of mass in ICRF and ITRF
- $T_{ICRF}^{ITRF}$ the $3 \times 3$ transformation matrix between ICRF and the ITRF.

The output quantities will be required at times other than those contained in the generated ephemeris file. Interpolation methods, such as those examined by Engelkemier [1992] provide the interpolation accuracy at a comparable to the numerical integration accuracy. With these parameters the ITRF position vector can be obtained as well by forming the product of the transformation matrix and the position vector in ICRF.
6.0 IMPLEMENTATION CONSIDERATIONS

6.1 Overview

During the first 30-120 days after launch, the ICESat spacecraft will be operated in a calibration orbit, with an 8-day repeat ground-track interval and 94-degree inclination. After this Verification Phase, it will be transitioned to a neighboring mission orbit at the same inclination, with a 183-day repeating ground track. To reduce the effect of the geopotential model errors on ICESat, which is the major source of error for ICESat POD, the gravity model improvement effort will be made through gravity tuning. Solar activity is predicted to peak shortly after launch, and decline significantly during the mission. The level of this activity correlates directly with the magnitude of atmospheric drag effects on the satellite.

The ICESat spacecraft will operate in two attitude modes depending on the angular distance between the orbit plane and the Sun (β’ angle). As shown in Figure 1, for low-β’ periods, such as that immediately following launch, the so-called "airplane-mode" is in use, with the solar panels perpendicular to the orbit plane. When the β’ angle exceeds 32 degrees, however a yaw maneuver places the satellite in the "sailboat-mode", with the axis of solar panels now in the orbit plane. While the two attitudes ensure that the solar arrays produce sufficient power year-round for bus and instrument operations, they introduce significantly different atmospheric drag effects due to the difference in cross-sectional area perpendicular to the velocity vector. The combination of high solar flux and low β’ angle at the start of the mission poses special challenges for POD and gravity tuning.
a) "airplane mode" for low $\beta'$

b) "sailboat mode" for high $\beta'$

Figure 1. ICESat Operational Attitudes
In this chapter, some considerations for implementing ICESat/GLAS POD algorithms are discussed. Section 6.2 summarizes the standard models to be used for ICESat/GLAS POD. These models include the reference system, the force models, and the measurement models. Section 6.3 describes the necessary input files which define the model parameters. Section 6.4 discusses some POD processing assumptions and issues. Section 6.5 describes the POD products. Section 6.6 discusses the expected ICESat/GLAS orbit accuracy based on the simulation study. Section 6.7 considers computational aspects. Section 6.8 describes the methods by which the accuracy of ICESat/GLAS POD products will be validated, Section 6.9 describes the POD processing timeline and activities, and Section 6.10 discusses about the POD reprocessing.

6.2 Standards

The standard models for the reference system, the force models and the measurement models to be used for the ICESat/GLAS POD are described in Table 6.1. These standards are based on the International Earth Rotation Service (IERS) standards [McCarthy, 1996], and the T/P standards [Tapley et al., 1994].

The gravity model to be used in the immediate post-launch period will be "best" available at launch, such as JGM-3 [Tapley et al., 1996] or EGM-96 [Lemoine et al., 1996]. As further gravity model improvements are made from other projects, such as GRACE, they will be incorporated for ICESat POD. At this writing, further study is required for the selection of the at-launch gravity model.
<table>
<thead>
<tr>
<th>Model</th>
<th>ICESat/GLAS Standard</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reference Frame</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conventional inertial system</td>
<td>ICRF</td>
<td>IERS</td>
</tr>
<tr>
<td>Precession</td>
<td>1976 IAU</td>
<td>IERS</td>
</tr>
<tr>
<td>Nutation</td>
<td>1980 IAU</td>
<td>IERS</td>
</tr>
<tr>
<td>Planetary ephemerides</td>
<td>JPL DE-200</td>
<td></td>
</tr>
<tr>
<td>Polar Motion</td>
<td>IERS</td>
<td></td>
</tr>
<tr>
<td>UT1-TAI</td>
<td>IERS</td>
<td></td>
</tr>
<tr>
<td>Station Coordinates</td>
<td>ITRF</td>
<td></td>
</tr>
<tr>
<td>Plate motion</td>
<td>Nuvel (NNR)</td>
<td>IERS</td>
</tr>
<tr>
<td>Reference ellipsoid</td>
<td>$a_e = 6378136.3$ m</td>
<td>Wakker [1990]</td>
</tr>
<tr>
<td></td>
<td>$1/f = 298.257$</td>
<td></td>
</tr>
</tbody>
</table>

| Force Models | | |
| GM | $398600.4415 \ km^3/\ell^2$ | Ries et al. [1992] |
| Geopotential | JGM-3 or EGM-96 | Tapley et al. [1996] |
| | | Lemoine et al. [1996] |
| $\overline{C}_{21}, \overline{S}_{21}$ – mean values | $\overline{C}_{21} = -0.187 \times 10^{-9}$ | |
| | $\overline{S}_{21} = +1.195 \times 10^{-9}$ | |
| $\overline{C}_{21}, \overline{S}_{21}$ – rates | $\overline{\dot{C}}_{21} = -1.3 \times 10^{-11} \ell/yr$ | (see rotational deformation) |
| | $\overline{\dot{S}}_{21} = +1.1 \times 10^{-11} \ell/yr$ | |
| epoch 1986.0 | | |
| Zonal rates | $J_2 = -2.6 \times 10^{-11} \ell/yr$ | Nerem et al. [1993] |
| | epoch 1986.0 | |
| N body | JPL DE405 | IERS |
| Indirect oblateness | point mass Moon on Earth $J_2$ | |
| Solid Earth tides | $k_2 = 0.3$ | IERS–Wahr [1981] |
| Frequency independent | $k_3 = 0.093$ | |
| Frequency dependent | | Wahl's theory |
| Ocean tides | CSR TOPEX 3.0 | Eanes and Bettadpur [1995] |
| Rotational deformation | $\Delta \overline{C}_{21} = -1.3 \times 10^9 (x_p - \overline{x}_p)$ | Nerem et al. [1994] |
| | $\Delta \overline{S}_{21} = +1.3 \times 10^9 (x_p - \overline{x}_p)$ | |
| based on $k_2/k_0 = 0.319$ | $\overline{x}_p = 0^\circ.046, \overline{y}_p = 0^\circ.294$ | |
| $\overline{x}_p = 0^\circ.0033/\ell/yr$ | | |
| $\overline{y}_p = 0^\circ.0026/\ell/yr$, epoch 1986.0 | | |
| Relativity | | |
| Solar radiation | solar constant = $14.560 \times 10^{-6}$ N/m$^2$ at 1 AU, conical shadow model for Earth and Moon | Ries et al. [1991] |
| | $R_e = 6402$ km, | |
| | $R_m = 1738$ km, | |
| | $R_s = 696,000$ km | |
| Atmospheric drag | density temperature model, daily flux and 3-hour constant $k_p$, | Barlier et al. [1977] |
3-hour lag for \( k_p \); 
1-day lag for \( f_{10.7} \); 
\( f_{10.7} \) average of previous 81 days 

<table>
<thead>
<tr>
<th>Earth radiation pressure</th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>Albedo and infrared second-degree zonal model, ( R_e = 6,378.136.3 )</td>
</tr>
<tr>
<td>Satellite parameters</td>
<td>ICESat/GLAS models</td>
</tr>
<tr>
<td></td>
<td>Box-wing model</td>
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</tbody>
</table>

**Measurement Models**

<table>
<thead>
<tr>
<th>Laser range</th>
<th>Marini &amp; Murray [1973]</th>
<th>IERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Troposphere</td>
<td></td>
<td>IERS</td>
</tr>
<tr>
<td>Relativity correction</td>
<td></td>
<td>IERS</td>
</tr>
<tr>
<td>Center of Mass/phase center</td>
<td></td>
<td>ICESat/GLAS model</td>
</tr>
<tr>
<td>GPS</td>
<td>MTT</td>
<td>Herring [1992]</td>
</tr>
<tr>
<td>Troposphere</td>
<td>dual frequency correction</td>
<td></td>
</tr>
<tr>
<td>Ionosphere</td>
<td>ICESat/GLAS model</td>
<td></td>
</tr>
<tr>
<td>Center of Mass/phase center</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relativity correction</td>
<td>applied</td>
<td></td>
</tr>
<tr>
<td>Site displacement</td>
<td>IERS</td>
<td></td>
</tr>
<tr>
<td>Induced permanent tide</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geometric tides</td>
<td>IERS</td>
<td></td>
</tr>
<tr>
<td>Frequency independent</td>
<td>( h_2 = 0.6090, ) ( l_2 = 0.0852, ) ( \delta = 0^\circ )</td>
<td></td>
</tr>
<tr>
<td>Frequency dependent</td>
<td>( K_1 ) IERS</td>
<td></td>
</tr>
<tr>
<td>Ocean loading</td>
<td>( h_2 = 0.6090, l_2 = 0.0852 ) with ( \bar{x}_p = 0^\circ.046, \bar{y}_p = 0^\circ.294 )</td>
<td></td>
</tr>
<tr>
<td>Rotational deformation</td>
<td>( \bar{x}_p = 0^\circ.0033/yr ) ( \bar{y}_p = 0^\circ.0026/yr, ) epoch 1986.0</td>
<td></td>
</tr>
</tbody>
</table>

Figures 2 and 3 show the ground station network for ICESat POD for GPS and SLR, respectively. Details of the adopted network may change at launch. Station coordinates will be adopted from the "best" available ITRF model, expected to be ITRF-99 or ITRF-2000. The ITRF model includes station velocities measured by space geodetic methods.
Figure 2. GPS Tracking Stations for ICESat POD

Figure 3. SLR Stations Tracking ICESat (20 degree Elevation Masks)
6.3 Ancillary Inputs

Some model parameters need to be supplemented as input files. Those include the Earth orientation parameters, such as $x_p$, $y_p$, and UT1, planetary ephemerides, geopotential parameters, ocean tides parameters, and solar flux data. In addition, information about the spacecraft attitude is required for the box-wing spacecraft model in the computation of nongravitational forces and to provide the correction for the GPS phase center location with respect to the spacecraft center of mass. The real-time attitude obtained during flight operations is adequate for this information, but will be checked against the precise attitude. Also, the GPS data from the IGS ground network are needed.

6.4 POD Processing Assumptions and Issues

Several assumptions were made for the POD processing. It assumes: 1) availability of continued operation of IGS GPS network and the SLR network, 2) IGS GPS data is available in Rinex format, 3) ICESat GPS receiver has performance characteristics of a TurboRogue, and ICESat GPS data is available in Rinex format, and 4) most relevant IGS, SLR and ICESat data is available within 24-36 hours. There are several issues for POD processing. Those include: 1) identification of problem GPS satellites, 2) identification of problems with ground station data, 3) processing arc length and relation to orbit maneuvers, and 4) handling of expected outgassing during early mission phase. For the July 2001 launch, orbit maneuvers are expected to occur as frequently as 5 days because of solar activity [Demarest, 1999]. These thrust forces
will not be modeled in the POD processing, but the maneuver times will be used for selection of the POD arc length.

6.5 POD Products

Two types of POD products will be generated: the Rapid Reference Orbit (RRO) and the operational POD. The former product will be generated within 12-24 hours for primarily internal use of assessing the operational orbit and support for mission planning, including orbit prediction using a subset of available flight data, which may consist of only small (~10 min) portions of the orbit once or twice per revolution. The operational POD will be generated within 7 days, possibly within 3 days, after accounting for problems identified in RRO (GPS satellite problems) and problems reported by IGS. This product will be used in generating the altimetry standard data products, particularly GLA06, the surface elevation.

6.6 Accuracy

The GPS tracking system has demonstrated its capability of providing high precision POD products through the GPS flight experiment on TOPEX/Poseidon [Melbourne et al., 1994]. Precise orbits computed from the GPS tracking data [Yunck et al., 1994; Christensen et al., 1994; Schutz et al., 1994] are estimated to have a radial orbit accuracy comparable to or better than the precise orbit ephemerides (POE) computed from the combined SLR and DORIS tracking data [Tapley et al., 1994]. When the reduced-dynamic orbit determination technique was employed with the GPS
data, which includes process noise accelerations that absorb dynamic model errors after fixing all dynamic model parameters from the fully dynamic approach, there is evidence to suggest that the radial orbit accuracy is better than 3 cm [Bertiger et al., 1994].

Previous simulation study [Rim et al., 1996] indicated that the ICESat/GLAS POD requirements could be met at 700 km altitude by either the gravity tuning or employing frequent estimation of empirical parameters, such as 1-rev one-cycle-per-revolution parameters, with fully dynamic approach. Because the mission orbit altitude was lowered to 600 km, and the satellite design has been changed since the study, a new in-depth simulation study [Rim et al., 1999] was conducted. It also indicates that even at 600 km altitude with maximum solar activity, the 5 cm and 20 cm radial and horizontal ICESat orbit determination requirement can be met with the gravity tuning effort.

### 6.7 Computational: CPU, Memory and Disk Storage

Table 6.2 compares the computational requirements for processing a typical one-day arc from a 24 ground station network with 30 sec sampling time for both T/P and ICESat/GLAS. These results are based on MSODP1 implemented on the Cray J90 and the HP-735/125.
Table 6.2  Computational Requirements for T/P and ICESat/GLAS POD using MSODP1: One-day Arcs with 24 Ground Stations

<table>
<thead>
<tr>
<th>Platform</th>
<th>Satellite</th>
<th>CPU (min)</th>
<th>Memory (Mw)</th>
<th>Disk* (Mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cray J90</td>
<td>T/P</td>
<td>20</td>
<td>2</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>ICESat/GLAS</td>
<td>40</td>
<td>2.5</td>
<td>59</td>
</tr>
<tr>
<td>HP-735</td>
<td>T/P</td>
<td>30</td>
<td>2</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>ICESat/GLAS</td>
<td>105</td>
<td>2.5</td>
<td>63</td>
</tr>
</tbody>
</table>

6.8  Product Validation

To validate the accuracy of ICESat/GLAS POD products, several methods would be employed. For the internal evaluation of the orbit accuracy, the orbit overlap statistics will be analyzed. Also, the data fit RMS value can be an indicator for the orbit quality. Comparisons between the orbits from different software, such as MSODP1 and GIPSY-OASIS, would serve as a valuable tool to assess the orbit accuracy. Results of Davis [1996] show that both MSODP1 and GIPSY-OASIS yield comparable results in a high fidelity simulation. Since the ICESat/GLAS will carry the laser reflector on board, the SLR data can be used as an independent data set to determine the ICESat/GLAS orbit. However, this approach assumes reasonably good tracking of the ICESat/GLAS orbit from the SLR stations. If insufficient data from the SLR network are available, the SLR data will be used to directly evaluate the GPS-determined orbit. Data fits for high elevation SLR passes can be used to evaluate the orbit accuracy of the

* This includes all necessary files (input files, data files, output files, executable file, temporary files).
ICESat/GLAS. The laser altimeter data will be used to assess the validation, however, this assessment can be accomplished only if the calibration and verification of the instrument have been accomplished.

6.9 POD Processing Timeline

During the first 30-120 days after launch, which is the Verification/Validation period, POD processing will tune the model parameters, including the gravity, and define adopted parameter set for processing Cycle 1. During the 183 days of the first Cycle, POD processing will assess and possibly further improve or refine parameters, such as assess GRACE gravity field, if available, and adopt new parameter set for processing of Cycle 2. POD processing will conduct ongoing assessment of POD quality after Cycle 1, and new parameter adoptions should be minimized and timed to occur at cycle boundaries.

6.10 POD Reprocessing

To produce improved orbits, reprocessing of data will be performed as often as annually. Any improvement in the model parameters will be adopted for the reprocessing.
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