Chapter 1

Fourier Series And Transforms

1.1 Fourier Series

A function that is defined and square-integrable over an interval, \([0,T]\), and is then periodically extended over the entire real line can be expressed as an infinite series of sines and cosines. This fact will be assumed without proof for now. Thus, suppose \(\tilde{g}(t)\) is a function that is periodic with period \(T\). Periodicity implies that the function is completely defined by its values over just the single interval, \([0,T]\). We may express \(\tilde{g}(t)\) as the following series:

\[
\tilde{g}(t) = a_0 + 2 \sum_{k=1}^{\infty} a_k \cos \frac{2\pi}{T} kt + 2 \sum_{k=1}^{\infty} b_k \sin \frac{2\pi}{T} kt,
\]

(1.1)

where \(a_k\) and \(b_k\) are constant coefficients. An example, where most of the coefficients are equal to zero, is given in Figure 1.1. The representation (1.1) of \(\tilde{g}(t)\) is called a Fourier series. Note that \(\tilde{g}(t)\) written this way is, in fact, periodic with period \(T\):

\[
\tilde{g}(t + T) = \tilde{g}(t), \quad \text{also} \quad \tilde{g}(t \pm nT) = \tilde{g}(t), \quad n = 0,1,2,\ldots.
\]

(1.2)

The proof of this is left to the reader.

Since spectral analysis originated in the theory of communications with electromagnetic signals, the function \(g\), whether periodic or not, often refers to a signal varying with respect to time, and the domain of definition of \(g\), i.e., \(-\infty < t < \infty\), is usually called the time domain. In geodesy, and many other fields, the domain does not have to be time, but could be distance or several spatial dimensions; in general, it may be any other independent variable: for spatial variables, the domain is called the space domain. At present, we assume that the domain is one
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The expression (1.1) for $\tilde{g}(t)$ in terms of a Fourier series involves quantities, $kT$. These are the harmonic frequencies of the signal. The fundamental frequency is $1/T$ ($k=1$). Again, the term “frequency” is generally associated with time signals and has units of cycle per unit of time, e.g., cycle per second, or Hz (Hertz). Frequency is the inverse of wavelength – high frequency (large $k$) implies short wavelength; low frequency (small $k$) implies long wavelength. This can be visualized (left to reader) by examining a graph of

$$\tilde{g}(t) = -0.5 + 2 \sum_{k=1}^{7} a_k \cos \frac{2\pi}{10} kt + 2 \sum_{k=1}^{3} b_k \sin \frac{2\pi}{10} kt,$$

where

$$\{a_k\} = \{1, -2, -0.5, 3.2, 2, -3, 0.7\} \quad \text{and} \quad \{b_k\} = \{2, -1.4, 3\}$$

by varying the parameter, $k$, ("sweeping" through frequencies). In geodesy, where $t$ is in the space domain, we still use the term frequency, although wavenumber (referring to the integers $k$) is also used. In this case, the units of frequency are cycle per unit of distance, e.g., [cy/m].

In case the frequency takes on continuous real values, we denote it by $f$ (cycle-frequency) having the same units as $k/T$, or also, $\omega$ (radian-frequency)

$$\omega = 2\pi f,$$
which has units [rad/s], or [rad/m], as the case may be. We will retain the cyclical frequency, although it means that we always have to include \( 2\pi \) [rad/cy], since the arguments of sinusoidal functions must be unitless (or in radians).

Sometimes, the Fourier series is written as

\[
\tilde{g}(t) = \sum_{k=0}^{\infty} a'_k \cos \frac{2\pi}{T} kt + \sum_{k=0}^{\infty} b'_k \sin \frac{2\pi}{T} kt ,
\]

or, even more compactly as

\[
\tilde{g}(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_k e^{\frac{i2\pi kt}{T}} ,
\]

where \( i^2 = -1 \), \( i \) is the *imaginary unit*, and the coefficients \( G_k \) are (in general cases) complex numbers. The relationship between the coefficients \( \{a_k, b_k\} \) in equation (1.1) and \( \{a'_k, b'_k\} \) in equation (1.5) is obvious. Since

\[
\cos \frac{2\pi}{T} kt = \frac{1}{2} \left( e^{-\frac{i2\pi kt}{T}} + e^{\frac{i2\pi kt}{T}} \right) \quad \text{and} \quad \sin \frac{2\pi}{T} kt = \frac{i}{2} \left( e^{-\frac{i2\pi kt}{T}} - e^{\frac{i2\pi kt}{T}} \right),
\]

one can easily derive the corresponding relationship between the coefficients \( \{a_k, b_k\} \) in equation (1.1) and \( \{G_k\} \) in equation (1.6):

\[
G_k = \begin{cases} 
  a_k - ib_k, & k > 0; \\
  a_0, & k = 0; \\
  a_{-k} + ib_{-k}, & k < 0.
\end{cases}
\]

The inverse relationship is given by

\[
a_k = \frac{1}{2T} (G_k + G_{-k}) , \quad k > 0; \\
a_0 = \frac{1}{T} G_0 , \quad k = 0; \\
b_k = \frac{i}{2T} (G_k - G_{-k}) , \quad k > 0.
\]

We pre-multiply the sum in equation (1.6) by \( 1/T \) to be consistent with the units used later for transforms of non-periodic functions. Thus, the units of \( G_k \) are the units of \( \tilde{g}(t) \) per frequency unit, since the units of \( T \) are inverse to those of frequency. For convenience of notation, we will
use only the complex-coefficient Fourier series (1.6), noting that with equations (1.8) and (1.9) one could always revert to the series (1.1) in terms of the sine and cosine functions.

The set of coefficients, \{a_k, b_k\} or \{G_k\}, is known as the (Fourier) spectrum of g. Given g, one can compute its spectrum using the orthogonality of the sines and cosines on the interval [0,T]. For the complex exponential, there is the following orthogonality relationship:

\[
\int_0^T e^{\frac{2\pi i k t}{T}} e^{-\frac{2\pi i \ell t}{T}} dt = \int_0^T \cos\left(\frac{2\pi}{T}(k-\ell) t\right) dt + i \int_0^T \sin\left(\frac{2\pi}{T}(k-\ell) t\right) dt
\]

\[
= \begin{cases} 
0, & k \neq \ell; \\
T, & k = \ell.
\end{cases}
\]

Therefore, multiplying equation (1.6) on both sides by \( e^{-\frac{2\pi i \ell t}{T}} \) and integrating yields, in view of equation (1.10),

\[
G_k = \int_0^T \tilde{g}(t) e^{-\frac{2\pi i \ell t}{T}} dt.
\]

Thus, given \( \hat{g}(t) \) in the interval, [0,T], one can find its spectrum according to equation (1.11); and, given its spectrum, one can compute \( \hat{g}(t) \) for any \( t \) using equation (1.6). The spectrum and its function are dual representations of the same information - one is equivalent to the other (as long as the function is continuous). The spectrum of a function displays the same information, but in a different domain – the frequency domain, where it is often more useful than in the time- (or space-) domain. The two relationships, (1.6) and (1.11), constitute a Fourier Transform pair for periodic functions. We will denote this relationship as

\[
\hat{g}(t) \leftrightarrow G_k.
\]

We will use this terminology and notation for other types for Fourier transforms, as well. Indeed, one can define Fourier transforms for non-periodic function under appropriate conditions, as well as for functions defined only at discrete points in the time or space domain. Therefore, to be more specific in the case above, we should say that it is the Fourier series transform pair.

1.2 Properties of Fourier Series

The Fourier series transform pair (1.12) may be manipulated using several linear operations whose results are summarized with the following (non-exhaustive list of) properties. They can be proved with relative ease from the basic transform pair, (1.12), and the definitions (1.6) and
It is noted that while we generally use only real-valued functions, \( g \), all definitions and properties hold equally for complex-valued functions and are written as such.

0. \( \tilde{g} (t) \leftrightarrow G_k \); \hfill (1.13)

1. \( a \tilde{g} (t) \leftrightarrow aG_k \): proportionality; \hfill (1.14)

2. \( \tilde{g}_1 (t) + \tilde{g}_2 (t) \leftrightarrow (G_1)_k + (G_2)_k \): superposition; \hfill (1.15)
   
   provided \( \tilde{g}_1 \) and \( \tilde{g}_2 \) have the same period, \( T \);

3. \( \tilde{g} (-t) \leftrightarrow G_{-k} \): symmetry; \hfill (1.16)

4. \( \tilde{g} (t - t_0) \leftrightarrow G_k e^{-\frac{2\pi i k t_0}{T}} \): translation; \hfill (1.17)

5. \( \frac{d}{dt} \tilde{g} (t) \leftrightarrow \frac{2\pi i k}{T} G_k \) differentiation in time. \hfill (1.18)

There is also an important theorem known as Parseval’s Theorem, which states that

6. \[
   \int_0^T \tilde{g}_1 (t) \tilde{g}_2 (t) dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} \overline{(G_1)_k (G_2^*)_k} \hfill (1.19)
   
   \] provided \( \tilde{g}_1 \) and \( \tilde{g}_2 \) have the same period, \( T \) (* denotes complex conjugate). The proof of this is easily done by substituting equation (1.6) on the left side, as follows:

\[
   \frac{1}{T^2} \int_0^T \left( \sum_{k=-\infty}^{\infty} (G_1)_k e^{\frac{2\pi i k t}{T}} \right) \overline{\left( \sum_{k=-\infty}^{\infty} (G_2^*)_k e^{-\frac{2\pi i k t}{T}} \right)} \ dt = \frac{1}{T^2} \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \overline{(G_1)_k (G_2^*)_k} \int_0^T e^{\frac{2\pi i (k-k') t}{T}} \ dt . \hfill (1.20)
   
   
   \]

With the orthogonality of complex exponentials, equation (1.10), the resulting equation (1.19) follows immediately.

### 1.3 The Fourier Transform

We now find corresponding Fourier transform pairs for certain non-periodic functions. Specifically, we restrict ourselves to square-integrable functions defined on the real line, which are functions, \( g \), such that
\[ \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty, \quad (1.21) \]

where \( |g(t)|^2 = g(t) g^*(t) \). Then its (continuous) Fourier transform is defined by (compare this with equation (1.11)):

\[ \mathcal{F}(g) \equiv G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} \, dt. \quad (1.22) \]

It can be shown that \( \mathcal{F}(g) \) is also square-integrable. Because of the square-integrability, it is said that \( g \) and \( G \) each has finite energy. In addition, the inverse (continuous) Fourier transform is given by

\[ \mathcal{F}^{-1}(G) \equiv g(t) = \int_{-\infty}^{\infty} G(f) e^{i2\pi ft} \, df. \quad (1.23) \]

\( G(f) \) is also known as the spectrum (or spectral density) of \( g \).

In general, \( G \) is complex, even if \( g \) is real (again, we will deal only with real functions, \( g \)). The spectrum can be decomposed into the amplitude spectrum:

\[ A(f) = \left( \left( \text{Re} G(f) \right)^2 + \left( \text{Im} G(f) \right)^2 \right)^{1/2}, \quad (1.24) \]

and the phase spectrum:

\[ \phi(f) = \tan^{-1} \frac{\text{Im} G(f)}{\text{Re} G(f)}. \quad (1.25) \]

Amplitude and phase together yield the spectrum in the form:

\[ G(f) = A(f) e^{i\phi(f)} \quad (1.26) \]

The square of the amplitude spectrum is also known as the energy spectrum.

If \( g \) has units [mgal], and \( t \) has units [m], then the spectrum, \( G \), has units \([\text{mgal/(cy/m)}]\), whence its alternative name, spectral density. The units of the Fourier series coefficients \( G_k \), equation (1.11), are consistent with this. Note that equations (1.22) and (1.23) could be written with radian frequency, \( \omega \), where from equation (1.4), \( df = d\omega / 2\pi \).

Finally, we see that periodic functions are not included in the definition of the Fourier transform (1.22), because they are not square-integrable – they have infinite energy! Although
we can get around this, later, in a formal sense, we note the practical distinction in Fourier 
transforms for periodic and non-periodic, finite-energy signal. Periodic (continuous) signals are 
represented by Fourier series; and, non-periodic, finite-energy (continuous) signals are 
represented by Fourier integrals.

1.4 Properties of Fourier Transforms

As with Fourier series, there are a number of properties associated with the Fourier transform of 
a non-periodic, finite-energy function and of linear operations performed on such a function. 
These are listed as follows for the (continuous) Fourier transform pair:

0. \( g(t) \leftrightarrow G(f) \); \hspace{1cm} (1.27)

1. \( a g(t) \leftrightarrow a G(f) \): \hspace{1cm} proportionality; \hspace{1cm} (1.28)

2. \( g_1(t) + g_2(t) \leftrightarrow G_1(f) + G_2(f) \): \hspace{1cm} superposition; \hspace{1cm} (1.29)

3. \( g(-t) \leftrightarrow G(-f) \): \hspace{1cm} symmetry; \hspace{1cm} (1.30)

4. \( G(t) \leftrightarrow g(-f) \): \hspace{1cm} duality; \hspace{1cm} (1.31)

5. \( g(at) \leftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right), \quad a \neq 0 \): \hspace{1cm} similarity theorem; \hspace{1cm} (1.32)

6. \( g(t-t_0) \leftrightarrow G(f)e^{-i2\pi ft_0} \): \hspace{1cm} translation; \hspace{1cm} (1.33)

7. \( \frac{d}{dt} g(t) \leftrightarrow i2\pi f G(f) \): \hspace{1cm} differentiation in time; \hspace{1cm} (1.34)

8. \( -it g(t) \leftrightarrow \frac{1}{2\pi} \frac{d}{df} G(f) \): \hspace{1cm} differentiation in frequency; \hspace{1cm} (1.35)

9. \( \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df \): \hspace{1cm} Parseval’s Theorem. \hspace{1cm} (1.36)

There are several other properties; some of these will be noted later on. Parseval’s Theorem 
will be proved in Section 2.1 on the basis of the Convolution Theorem. A proof of Property 5, 
equation (1.32), is given below.
Proof: Let \( a \) be a non-zero constant; and let \( t' = at \). Then

\[
\mathcal{F}(g(at)) = \frac{1}{|a|} \int_{-\infty}^{\infty} g(t') e^{-\frac{i2\pi f}{a} dt'} = \frac{1}{|a|} G\left(\frac{f}{a}\right)
\]

Example: We can use Property 5 to determine the Fourier transform pair of a function if we change units through a change of variables, such as a change from spatial to temporal domains. Usually this is needed if a velocity, \( v \), is involved, and we wish to change from the time domain \((t)\) to the distance domain \((s)\), and vice versa. Assuming the velocity is constant, the change of variables is given by

\[
s = vt.
\]

Then if \( G(\mu) \) is the spectrum of \( g(s) \), the spectrum, \( G'(f) \), of \( g'(t) = g(vt) \) is given by \( G'(f) = G(f/v)/|v| \). Suppose \( g \) has units [mgal] and \( s \) has units [m], then \( G(\mu) \) has units [mgal/(cy/m)]. Now if \( v \) has units [m/s], then \( t \) has units [s], \( f \) has units [cy/s], \( g'(t) \) still has units [mgal], and its spectrum, \( G'(f) \), has units [mgal/(cy/s)].

1.5 Some Important Examples

The following examples of Fourier transforms of particular functions will be very useful.

Example 1: The rectangular ("box-car") function (Figure 1.2). Let

\[
b_1(t) = \begin{cases} 
1, & |t| < \frac{1}{2}; \\
0.5, & |t| = \frac{1}{2}; \\
0, & |t| > \frac{1}{2}.
\end{cases}
\]

Then the Fourier transform, according to the definition (1.22) is given by
\[ B_1(f) = \mathcal{F} \left( b_1(t) \right) = \int_{-\infty}^{\infty} b_1(t) e^{-i2\pi ft} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi ft} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi ft) dt \]
\[ = \frac{\sin(\pi f)}{\pi f} \]  

(1.40)

We define the value of the rectangular function at \(|t|=1/2\) even though it is of no consequence for the direct Fourier transform (1.40). However, the inverse Fourier transform yields

\[ \mathcal{F}^{-1}(B_1(f)) \big|_{t=\pm1/2} = \int_{-\infty}^{\infty} B_1(f) e^{i2\pi f \left( \pm \frac{1}{2} \right)} df = 2 \int_{0}^{\infty} \frac{\sin(\pi f)}{\pi f} \cos(\pi f) df = \frac{1}{2}, \]  

(1.41)

which agrees with the defined value.

Figure 1.2: Rectangular function and its Fourier transform, the sinc function.

The Fourier transform of the rectangular function is the so-called "sinc" function of \(f\) (formally the “sine cardinal” function):

\[ \text{sinc}(f) = \frac{\sin(\pi f)}{\pi f}. \]  

(1.42)

Because this function appears so frequently, it is given this special name. Note that since the inverse Fourier transform of the sinc function is the rectangular function, we have with equation (1.23) and \(t = 0\):
\begin{equation}
\int_{-\infty}^{\infty} \text{sinc}(f) \, df = 1.
\end{equation}

That is, the area enclosed by the sinc function in the frequency domain is also unity, as it is for the box-car function in the time domain.

By the similarity theorem (1.32) and proportionality (1.28), we have

\begin{equation}
\frac{1}{T} b_1 \left( \frac{t}{T} \right) \leftrightarrow \frac{\sin(\pi Tf)}{\pi Tf} = B_1(Tf).
\end{equation}

From this we see that as the rectangular function’s base shrinks ($T$ decreases), the main “lobes” of the sinc function expand (the first zero of the sinc function is at $f = \pm 1/T$); the opposite clearly holds: as $T$ increases, the lobes of the sinc function become narrower.

**Example 2:** The *Dirac* function (also known as the *delta* function, or the *impulse* function). This is not really a function in the sense that it is not well defined for all $t$. Specifically, the Dirac function, $\delta(t)$, is the "function" that satisfies the following:

a) \( \delta(t) = 0, \) for all \( t \neq 0 \); \hspace{1cm} (1.45)

b) \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1; \) \hspace{1cm} (1.46)

c) \( \int_{-\infty}^{\infty} \delta(t-t_0) g(t) \, dt = \int_{-\infty}^{\infty} \delta(t_0-t) g(t) \, dt = g(t_0). \) \hspace{1cm} (1.47)

In order for the area under the delta function to be non-zero, according to equation (1.46), even though it is zero almost everywhere, in view of equation (1.45), its "value" at $t = 0$ must be infinite. This can be seen by using the following approximate forms of the delta function. Consider the functions $b_1(t/T)/T$ with $0 < T \leq 1$ (see Figure 1.3). Note that the area under each of these scaled box-car functions is equal to 1. As $T$ approaches zero, the magnitude of $b_1(t/T)/T$ at $t = 0$ approaches infinity, while it approaches zero everywhere else. Therefore, we have

\( \delta(t) = \lim_{T \to 0} \frac{1}{T} b_1 \left( \frac{t}{T} \right). \) \hspace{1cm} (1.48)

Again, this is just a formal expression because, of course, the limit does not exist for $t = 0$. 

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But, we can now (formally) determine the Fourier transform of the delta function by determining first the Fourier transform of the function $b_1(t/T)/T$ and then taking the limit as $T$ approaches zero. With equation (1.44) we have

$$\mathcal{F} \left( \delta \left( t \right) \right) = \lim_{T \to 0} \frac{1}{T} \mathcal{F} \left( b_1 \left( \frac{t}{T} \right) \right) = \lim_{T \to 0} \left( \text{sinc} \left( Tf \right) \right) \quad (1.49)$$

$$= 1, \quad \text{for all} \ t$$

That is, the Fourier transform of the delta function is a constant (equal to 1). The units of the delta function, $\delta \left( t \right)$, according to equation (1.46) are $1/(\text{units of} \ t)$, whatever the units of $t$.

![Figure 1.3: The functions $b_1(t/T)/T$ with $T \to 0$.](image)

We say that $\delta \left( t \right)$ and 1 constitute a Fourier transform pair in the limit. We will be able to manipulate the delta function and its transform according to the above methods that rigorously first use the rectangular functions, and then apply the appropriate limit. Here we always assume the interchangeability of the limiting process and whatever other operators occur, provided the result either exists or can be interpreted in terms of the delta functions.

**Example 3:** Fourier transform of a periodic function. When defining the Fourier transform, (1.22), for continuous functions, we specifically excluded periodic functions because they do not have finite energy. However, with the concept of the delta function, we can formally determine the Fourier transform of a periodic function. This is done by noting that from equation (1.49), $\mathcal{F} \left( \delta \left( t \right) \right) = 1$, we have $\mathcal{F}^{-1}(1) = \delta \left( t \right)$; or
\[ \delta(t) = \int_{-\infty}^{\infty} e^{i2\pi ft} df. \]  
\hspace{1cm} (1.50)

Now change variables in equation (1.50) as follows: first replace \( t \) by \( k/T \); subsequently replace \( f \) by \( -t \). Then

\[ \int_{-\infty}^{\infty} e^{-\frac{i2\pi kt}{T}} dt = \delta \left( \frac{k}{T} \right). \]  
\hspace{1cm} (1.51)

Now for periodic functions expressed as Fourier series, (1.6), the Fourier transform, formally, can be expressed as

\[ \mathcal{F} \left( \tilde{g}(t) \right) = G(f) = \int_{-\infty}^{\infty} \left( \frac{1}{T} \sum_{k=-\infty}^{\infty} G_k e^{\frac{i2\pi kT}{T}} \right) e^{-i2\pi ft} dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_k \int_{-\infty}^{\infty} e^{-i2\pi \left( f - \frac{k}{T} \right)} dt \]  
\hspace{1cm} (1.52)

That is, the Fourier transform of a periodic function with Fourier coefficients \( G_k \) is an infinite sequence of impulses scaled by \( G_k/T \) and spaced along the frequency axis at the discrete frequencies \( k/T \) (Figure 1.4). Note that the units of the delta function in (1.48) are 1/(units of frequency), canceling the units of \( T \); so, \( G(f) \) and \( G_k \) have the same units.

The Fourier transform of a periodic function, as expressed in equation (1.52), is formulated this way only to make it consistent with our previous definition of a Fourier transform of non-periodic and square-integrable functions (i.e., finite-energy functions). In that sense it has some usefulness, but mainly we will consider the Fourier series pair, equations (1.6) and (1.11), to be the appropriate Fourier (series) transform pair, as noted earlier, thus avoiding the use of the delta function.
Example 4. The sampling function. Consider the periodic function that is an infinite sequence of identical rectangular functions, each having $T$ as its base:

$$
\tilde{g}(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} b_k \left( \frac{t - k \Delta t}{T} \right),
$$

(1.53)

where $\Delta t > T$ is the spacing between them (Figure 1.5a), and hence the period of $\tilde{g}(t)$ is $\Delta t$. We define $s(t)$ to be the limit of $\tilde{g}(t)$ as $T \to 0$ while the area of each rectangle remains equal to 1. From equation (1.48), $s(t)$ is given by (see also Figure 1.5b)

$$
s(t) = \lim_{T \to 0} \tilde{g}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k \Delta t).
$$

(1.54)
Being periodic, with period, $\Delta t$, $\tilde{g}(t)$, given by equation (1.53), can be represented as a Fourier series (1.6), where the Fourier coefficients are given by equation (1.11), which is an integral over any period of $\tilde{g}(t)$. We choose $[-T/2, \Delta t-T/2]$:

$$G_k = \frac{1}{T} \int_{-T/2}^{N-T/2} b_1 \left( \frac{t}{T} \right) e^{-\frac{2\pi i k t}{\Delta t}} dt.$$  \hspace{1cm} (1.55)

Now, outside the integration interval the rectangular function, $b_1$, is zero, so the integration can be extended to $\pm\infty$, and with a suitable change of integration variable ($t' = t/T$), we get, using the result (1.40) and the definition (1.42),

$$G_k = \frac{1}{T} \int_{-\infty}^{\infty} b_1 \left( \frac{t}{T} \right) e^{-\frac{2\pi i k t}{\Delta t}} dt = \int_{-\infty}^{\infty} b_1 \left( t' \right) e^{-\frac{2\pi i k T}{\Delta t} t'} dt'$$

\hspace{1cm} = \text{sinc} \frac{kT}{\Delta t} \hspace{1cm} (1.56)

Formally, the Fourier transform of $\tilde{g}(t)$, according to equation (1.52) is an infinite sequence of impulses having magnitudes, $G_k/T$. Substituting equation (1.56) into the right side of equation (1.52), we have

$$\mathcal{F} \left( \tilde{g}(t) \right) = \frac{1}{\Delta t} \sum_{k=\infty}^{\infty} \left( \text{sinc} \frac{kT}{\Delta t} \right) \delta \left( f - \frac{k}{\Delta t} \right).$$  \hspace{1cm} (1.57)
Then we apply the limit as $T \to 0$; and, noting that $\lim_{T \to 0} \left( \text{sinc} \frac{kT}{\Delta t} \right) = 1$, we obtain with equation (1.54):

$$
\mathcal{F} \left( \sum_{k=-\infty}^{\infty} \delta (t - k\Delta t) \right) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} \delta \left( f - \frac{k}{\Delta t} \right).
$$

(1.58)

The infinite sequence of impulses, $s(t)$, being the limit of the sequence of rectangular functions, is called the sampling function, because if multiplied with some arbitrary function it samples the latter using impulses (see Chapter 3). The Fourier transform of the sampling function is, again, a sampling function, but now in the frequency domain.

### 1.6 2-D Fourier Series and Transforms

In geodesy and geophysics we deal with “signals” on higher dimensioned domains such as the plane, the sphere, and three-space. All the concepts of Fourier series and transforms easily carry over into these higher dimensions – especially if the underlying coordinate system is Cartesian.

The Fourier series of a function periodic in two (Cartesian) variables is given by

$$
\mathcal{F} \left( \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} G_{k_1,k_2} e^{i2\pi \left( \frac{k_1}{T_1} x_1 + \frac{k_2}{T_2} x_2 \right)} \right),
$$

(1.59)

where the coefficients are given by

$$
G_{k_1,k_2} = \frac{T_1 T_2}{2\pi} \int_{0}^{T_1} \int_{0}^{T_2} \mathcal{F} \left( \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} G_{k_1,k_2} e^{-i2\pi \left( \frac{k_1}{T_1} x_1 + \frac{k_2}{T_2} x_2 \right)} \right) dx_1 dx_2.
$$

(1.60)

We use the complex notation for convenience, but one could as well write equations (1.59) and (1.60) in terms of sines and cosines. Note that the periodicity of $g$ may be different in the two dimensions; viz., the periods are $T_1$ and $T_2$, such that $g \left( x_1 \pm n_1 T_1, x_2 \pm n_2 T_2 \right) = g \left( x_1, x_2 \right)$, for any integers, $n_1$ and $n_2$. Clearly, higher-dimensioned functions have analogous series expansions where the generalization should be obvious.

Assuming finite-energy functions in two dimensions, i.e.,

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x_1, x_2)|^2 dx_1 dx_2 < \infty,
$$

(1.61)

we have the (continuous) Fourier transform pair:
\[ G(f_1, f_2) = \int \int g(x_1, x_2) e^{-i2\pi (f_1 x_1 + f_2 x_2)} dx_1 dx_2, \]  
(1.62)

\[ g(x_1, x_2) = \int \int G(f_1, f_2) e^{i2\pi (f_1 x_1 + f_2 x_2)} df_1 df_2. \]  
(1.63)

If the units of \( g \) are [mgal] and those of \( x_1, x_2 \) are each [m], then \( G \) has units of [mgal/(cy/m)^2].

The properties for the 1-D Fourier transforms, equations (1.14) through (1.19) and (1.27) through (1.36), can naturally be extended for the 2-D transform. Only the following are mentioned in particular, starting with the transform pair:

0. \( g(x_1, x_2) \leftrightarrow G(f_1, f_2); \)  
(1.64)

1. \( g(a_1 x_1, a_2 x_2) \leftrightarrow \frac{1}{|a_1 a_2|} G\left(\frac{f_1}{a_1}, \frac{f_2}{a_2}\right), \quad a_1 \neq 0, \quad a_2 \neq 0: \) similarity theorem;  
(1.65)

2. \( \frac{\partial^p}{\partial x_1^p} \frac{\partial^q}{\partial x_1^q} g(x_1, x_2) \leftrightarrow (i2\pi f_1)^p (i2\pi f_2)^q G(f_1, f_2): \) differentiation;  
(1.66)

3. \[ \int \int g_1(x_1, x_2) g_2^*(x_1, x_2) dx_1 dx_2 = \int \int G_1^*(f_1, f_2) G_2(f_1, f_2) df_1 df_2: \]

Parseval’s Theorem.  
(1.67)

Parseval’s Theorem, again, will be proved later using the convolution theorem.

We can now also define the rectangular (box-car) function in two dimensions:

\[
\begin{align*}
&b_2 = \begin{cases} 
1, & |x_1| < \frac{1}{2} \text{ and } |x_2| < \frac{1}{2}; \\
0.5, & |x_1| = \frac{1}{2} \text{ and } |x_2| \leq \frac{1}{2} \text{ or } |x_1| \leq \frac{1}{2} \text{ and } |x_2| = \frac{1}{2}; \\
0, & |x_1| > \frac{1}{2} \text{ or } |x_2| > \frac{1}{2};
\end{cases} 
\end{align*}
\]  
(1.68)

and its Fourier transform is simply the product of sinc functions (the proof is left to the reader):

\[ \mathcal{F}\left(b_2(x_1, x_2)\right) = B_2(f_1, f_2) = \text{sinc}(f_1) \text{sinc}(f_2). \]  
(1.69)
Analogous to one dimension, the Dirac delta function is defined by

\begin{align}
\text{a) } \delta(x_1, x_2) &= 0, \text{ if } x_1 \neq 0 \text{ or } x_2 \neq 0; \quad (1.70) \\
\text{b) } \iiint_{-\infty}^{\infty} \delta(x_1, x_2) \, dx_1 \, dx_2 &= 1; \quad (1.71) \\
\text{c) } \iiint_{-\infty}^{\infty} \delta(x'_1 - x_1, x'_2 - x_2) \, g(x'_1, x'_2) \, dx'_1 \, dx'_2 &= g(x_1, x_2); \quad (1.72)
\end{align}

and its Fourier transform is

\[ \mathcal{F}\left(\delta(x_1, x_2)\right) = 1, \text{ for all } f_1, f_2 \quad (1.73) \]

### 1.7 2-D Transforms of the Potential Function

Consider a function defined in three-dimensional space, but restrict the Fourier transform to the first two dimensions with the third variable held fixed. For certain functions these 2-D transforms, with different values of the third variable, can be related to each other. This is the case, for example, for the gravitational potential, \( v \), that satisfy Laplace’s equation in the half space \( z > 0 \):

\[ \nabla^2 v = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2} \right) v = 0. \quad (1.74) \]

Solutions to this equation are said to be harmonic. We use the notation \( z \), rather than \( x_3 \), for the third dimension in order to emphasize that the Fourier transformations are restricted to the first two dimensions and \( z \) may be viewed more as a parameter. Since \( v \) is a Newtonian potential, we have \( v \to 0 \) as \( z \to \infty \). Let \( V \) be the 2-D Fourier transform of \( v \) with respect to \( x_1, x_2 \):

\[ V(f_1, f_2; z) = \mathcal{F}(v) = \iiint_{-\infty}^{\infty} v(x_1, x_2, z) e^{-i2\pi(f_1 x_1 + f_2 x_2)} \, dx_1 \, dx_2. \quad (1.75) \]

The inverse transform is given by

\[ v(x_1, x_2, z) = \mathcal{F}^{-1}(V) = \iiint_{-\infty}^{\infty} V(f_1, f_2; z) e^{i2\pi(f_1 x_1 + f_2 x_2)} \, df_1 \, df_2. \quad (1.76) \]
Applying equation (1.74) to (1.76) yields

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla^2 \left( V(f_1, f_2; z) e^{i2\pi(f_1x_1 + f_2x_2)} \right) df_1 df_2 = 0. \]  

(1.77)

Hence, performing the differentiation of the integrand according to equation (1.74), we get for all \( z > 0 \):

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\partial^2}{\partial z^2} V(f_1, f_2; z) - (2\pi)^2 \left( f_1^2 + f_2^2 \right) V(f_1, f_2; z) \right) e^{i2\pi(f_1x_1 + f_2x_2)} df_1 df_2 = 0. \]  

(1.78)

This has to hold for all (or for arbitrary) \( x_1, x_2 \), which implies that the integrand must be equal to zero, and we have the following differential equation for \( V \) in terms of the variable \( z \):

\[ \frac{\partial^2}{\partial z^2} V(f_1, f_2; z) - (2\pi)^2 \left( f_1^2 + f_2^2 \right) V(f_1, f_2; z) = 0. \]  

(1.79)

The solution to this equation is given by (as can easily be verified by back-substitution into equation (1.79))

\[ V(f_1, f_2; z) = C_1 e^{-2\pi z \sqrt{f_1^2 + f_2^2}} + C_2 e^{2\pi z \sqrt{f_1^2 + f_2^2}}, \]  

(1.80)

where \( C_1 \) and \( C_2 \) are constants. We must have \( C_2 = 0 \); otherwise \( v \to \infty \) as \( z \to \infty \), which contradicts our assumption that \( v \) is a Newtonian potential. Also, suppose that on the plane \( z = 0 \) we have

\[ V(f_1, f_2; 0) = V_0(f_1, f_2) \]  

(1.81)

Then, finally, the solution (1.80) is given by

\[ V(f_1, f_2; z) = V_0(f_1, f_2) e^{-2\pi zf}, \]  

(1.82)

where we define the radial frequency

\[ f = \sqrt{f_1^2 + f_2^2}. \]  

(1.83)

The 2-D spectra of \( v \) at two levels, \( z_1 \) and \( z_2 \), are thus related according to equation (1.82) by the factor \( e^{-2\pi (z_1 - z_2)f} \).

The inverse Fourier transform (1.76) now becomes
\[ v(x_1, x_2, z) = \mathcal{F}^{-1}(V) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_0(f_1, f_2) e^{-2\pi i f_1 x_1 + 2\pi i f_2 x_2} df_1 df_2. \quad (1.84) \]

Then, for any \( n \geq 0 \)

\[ \frac{\partial^n}{\partial z^n} v(x_1, x_2, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-2\pi f)^n V_0(f_1, f_2) e^{-2\pi i f_1 x_1 + 2\pi i f_2 x_2} df_1 df_2, \quad (1.85) \]

which implies that the Fourier transform of the \( z \)-derivatives of \( v \) are given by

\[ \mathcal{F}\left( \frac{\partial^n}{\partial z^n} v(x_1, x_2, z) \right) = (-2\pi f)^n \mathcal{F}\left( v(x_1, x_2, z) \right). \quad (1.86) \]

Equation (1.86) holds only for those finite energy (in 2-D) functions satisfying our original assumptions of harmonicity and vanishing at infinity. Clearly, the spectrum of the derivatives is amplified at high frequencies (large \( n \)). This feature may be advantageous, as when we seek to detect the high-frequency components of the gravity field and thus use gravity gradiometry; or, it may be deleterious, as when one numerically differentiates a noisy potential estimate (and there is significant noise at the high frequencies).

### 1.8 The Hankel Transform

We now consider the special case of a function defined on the 2-D plane, but depending on just a single variable, the radial distance from the origin. Thus, let

\[ r = \sqrt{x_1^2 + x_2^2}, \quad (1.87) \]

and then suppose that

\[ g(x_1, x_2) = g(r). \quad (1.88) \]

The Fourier transform in this case is, according to equation (1.62)
\[ F \left( g(x_1, x_2) \right) = \iint_{-\infty}^{\infty} g(x_1, x_2) e^{-2\pi i (f_1 x_1 + f_2 x_2)} \, dx_1 \, dx_2 = \iint_{0}^{2\pi} g(r) e^{-2\pi i f r \cos(\phi - \alpha)} \, r \, d\phi \, dr \]

(1.89)

where \( x_1 = r \cos \phi \), \( x_2 = r \sin \phi \) and \( f_1 = f \cos \alpha \), \( f_2 = f \sin \alpha \). The integral in the last line of equation (1.89) with respect to \( \phi \) is known to be

\[ \int_{0}^{2\pi} e^{-2\pi i f r \cos(\phi - \alpha)} \, d\phi = 2\pi J_0(2\pi f r), \]

(1.90)

for any \( \alpha \), where \( J_0 \) is the zero-order Bessel function of the first kind. We have, therefore,

\[ G(f) = 2\pi \int_{0}^{\infty} r g(r) J_0(2\pi f r) \, dr, \]

(1.91)

which is called the Hankel transform of \( g \).

The Hankel transform is just a special case of the 2-D Fourier transform, where the function being transformed has circular symmetry, given by equation (1.88). We then have also the inverse Hankel transform, derived from the inverse Fourier transform in exactly the same way. The result is:

\[ g(r) = 2\pi \int_{0}^{\infty} f G(f) J_0(2\pi f r) \, df. \]

(1.92)

Note that although \( G(f) \) is a function of only one frequency (the radial frequency, \( f \), equation (1.83)), its units still contain the inverse square of frequency units. For example, if \( g \) has units [mgal] and \( r \) has units [m], then the units of \( G \) are [mgal/(cy/m)^2].

1.9 Properties of the Hankel Transform

Being a special case of the Fourier transform, there are similar properties for the Hankel transform. We start with the basic Hankel transform pair:

0. \( g(r) \leftrightarrow G(f) \); 

(1.93)
1. \( a g (r) \leftrightarrow aG (f) \): proportionality; (1.94)

2. \( g_1 (r) + g_2 (r) \leftrightarrow G_1 (f) + G_2 (f) \): superposition; (1.95)

3. \( G (r) \leftrightarrow g (f) \) duality; (1.96)

4. \( g (ar) \leftrightarrow \frac{1}{a^2} G \left( \frac{f}{a} \right) \), \( a > 0 \): similarity theorem; (1.97)

5. \( \int_0^\infty g_1 (r) g_2 (r) rdr = \int_0^\infty G_1 (f) G_2 (f) f df \): Parseval’s Theorem. (1.98)

Parseval’s Theorem in this case derives directly from the corresponding theorem for functions of two independent variables, equation.

The rectangular function in this case is given by the cylinder function, having circular symmetry:

\[
b_c (r) = \begin{cases} 1 \frac{1}{\pi a^2}, & r < a; \\ 0, & r > a; \end{cases} \tag{1.99}
\]

where \( a \) is the radius of the cylinder. The Hankel transform of \( b_c (r) \) is given by

\[
B_c (f) = 2\pi \int_0^a \frac{r}{\pi a^2} \int_0^{2\pi f r} J_0 (2\pi f r) dr
\]

\[
= \frac{2\pi}{(2\pi f)^2} \int_0^a \frac{2\pi f r}{\pi a^2} J_0 (2\pi f r) d(2\pi f r) = \frac{1}{\pi a^2 f (2\pi f)} \left[ 2\pi f r J_1 (2\pi f r) \right]^0_{a}
\]

\[
= \frac{1}{\pi af} J_1 (2\pi fa)
\]

where \( J_1 \) is the first-order Bessel function of the first kind, and \( \frac{d}{dx} (xJ_1 (x)) = xJ_0 (x) \).
1.10 Spherical Harmonic Series and the Legendre Transform

For a function defined on a sphere, we have coordinates \((\theta, \lambda)\) - geocentric co-latitude and longitude (we could also use geocentric latitude, \(\phi\), instead, where \(\phi = 90^\circ - \theta\)). We note that any such function is periodic in \(\lambda\) with period \(2\pi\) and (technically) periodic in \(\theta\) with period \(2\pi\) (although it is defined only for \(0 \leq \theta \leq \pi\)). Therefore, we expect that reasonably well behaved functions might be expressed as series (not integrals) of sinusoidal functions – in this case, however, in addition to the usual sinusoidal functions of longitude, the special geometry of the sphere calls for Legendre functions of \(\cos \theta\). We have the following well known expansion, where we omit the “~” notation used earlier for periodic functions because the arguments, \((\theta, \lambda)\), already signify periodicity:

\[
g(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} G_{n,m} \bar{Y}_{n,m}(\theta, \lambda),
\]

where the \textit{spherical harmonic functions} are given by

\[
\bar{Y}_{n,m}(\theta, \lambda) = \bar{P}_{n,|m|}(\cos \theta) \begin{cases} 
\cos m\lambda, & m \geq 0; \\
\sin |m|\lambda, & m < 0;
\end{cases}
\]

(1.102)

and where \(\bar{P}_{n,|m|}(\cos \theta)\) is the \textit{fully-normalized}, associated Legendre function of the first kind of degree \(n\) and order \(|m|\), with \(-n \leq m \leq n\) and \(n \geq 0\). This function is defined as follows:

\[
\bar{P}_{n,m}(\cos \theta) = \begin{cases} 
\sqrt{2n+1} P_{n,0}(\cos \theta), & m = 0; \\
\frac{2(2n+1)(n-m)!}{(n+m)!} P_{n,m}(\cos \theta), & m > 0;
\end{cases}
\]

(1.103)

where the un-normalized Legendre functions, \(P_{n,m}(\cos \theta)\), are given by

\[
P_{n,m}(\cos \theta) = \sin^m \theta \frac{d^m}{d(\cos \theta)^m} P_n(\cos \theta),
\]

(1.104)

and the Legendre polynomials, \(P_n = P_{n,0}\), are given by

\[
P_n(\cos \theta) = \frac{1}{2^n n!} \frac{d^n}{d(\cos \theta)^n} (\cos^2 \theta - 1)^n.
\]

(1.105)
The function, $\overline{P}_{n,m}$, is normalized such that the spherical harmonic functions, $Y_{n,m}$, which are orthogonal, are, in fact, orthonormal:

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \overline{P}_{n,m}(\theta, \lambda) Y_{p,q}(\theta, \lambda) \sin \theta d\theta d\lambda = \begin{cases} 0, & n \neq p \text{ or } m \neq q; \\ 1, & n = p \text{ and } m = q. \end{cases}$$  \hspace{1cm} (1.106)

It is readily shown using this orthogonality relationship that

$$G_{n,m} = \frac{1}{4\pi} \int_\sigma g(\theta, \lambda) \overline{P}_{n,m}(\theta, \lambda) d\sigma,$$  \hspace{1cm} (1.107)

where $d\sigma = \sin \theta d\theta d\lambda$, and $\sigma = \{(\theta, \lambda) | 0 \leq \theta \leq \pi, 0 \leq \lambda \leq 2\pi\}$ represents the surface of the unit sphere. Equation (1.107) is called the Legendre transform of $g$, or the Legendre spectrum of $g$. $G_{n,m}$ and $g(\theta, \lambda)$ are the Legendre transform pair. We will deal only with real functions, $g$; and the coefficients, $G_{n,m}$ are then all real, as well.

An alternative form of the Legendre transform pair may be found in the literature, although it is not as often used in geodesy:

$$g(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \gamma_{n,m} \overline{P}_{n,m}(\cos \theta) e^{im\lambda},$$  \hspace{1cm} (1.108)

$$\gamma_{n,m} = \frac{1}{4\pi \varepsilon_m} \int_0^{2\pi} \int_0^\pi g(\theta, \lambda) \overline{P}_{n,m}(\cos \theta) e^{-im\lambda} \sin \theta d\theta d\lambda,$$  \hspace{1cm} (1.109)

where

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases} \hspace{1cm} (1.110)$$

The relationship between the coefficients $\gamma_{n,m}$ and $G_{n,m}$ is given by

$$\gamma_{n,m} = \begin{cases} \frac{1}{2} \left( G_{n,m} - iG_{n,-m} \right), & m > 0; \\ G_{n,0}, & m = 0; \\ \frac{1}{2} \left( G_{n,-m} + iG_{n,m} \right), & m < 0. \end{cases}$$  \hspace{1cm} (1.111)
In this case, the Legendre spectrum, defined by $\gamma_{n,m}$, is complex, but it satisfies $\gamma_{n,m} = \gamma_{n,-m}^*$ if $g$ is a real function.

Many functions in geodesy are also defined in terms of just one variable, $\psi$, the spherical distance, that technically can be considered as an angle in the domain of a semi-circle: $\{\psi | 0 \leq \psi \leq \pi\}$. In this case, one could expand the function (if considered to be periodic) as a series of sinusoids as before. Or, to be consistent with the spherical harmonic expansion, one can also use Legendre polynomials of $\cos \psi$, since the Legendre polynomials, $P_n$, form a complete basis for functions defined on $[-1,1]$, or on $0 \leq \psi \leq \pi$. We have, therefore, for a function, $g$, defined on $0 \leq \psi \leq \pi$:

$$g(\psi) = \sum_{n=0}^{\infty} (2n+1) G_n P_n(\cos \psi),$$

(1.112)

and

$$G_n = \frac{1}{2} \int_0^\pi g(\psi) P_n(\cos \psi) \sin \psi \, d\psi,$$

(1.113)

which, together, give the one-dimensional Legendre transform pair. Here, $G_n$ is the Legendre spectrum; it is not exactly consistent in terms of scale with the spectrum of functions defined on the sphere, due to the normalization of $P_{n,0}$:

$$G_{n,0} = \sqrt{2n+1} G_n.$$

(1.114)

Although the above definition (1.113) is somewhat less cumbersome, it is not universally used in the literature; any differences concern the normalization.

### 1.11 Properties of Legendre Transforms

Some essential properties associated with the Legendre transform are analogous to properties of the Fourier transform. Those given for the 2-D Legendre transform hold equally for the 1-D transform on the basis of equation (1.114). Again, we assume the basic Legendre transform pair:

0. $g(\theta,\lambda) \leftrightarrow G_{n,m};$

1. $ag(\theta,\lambda) \leftrightarrow aG_{n,m};$ proportionality;

(1.115)

(1.116)
2. \( g_1(\theta, \lambda) + g_2(\theta, \lambda) \leftrightarrow (G_1)_{n,m} + (G_2)_{n,m} \): superposition; 

\[ (1.117) \]

3. \( \frac{1}{4\pi} \int_\sigma (g(\theta, \lambda))^2 \, d\sigma = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (G_{n,m})^2 \): Parseval’s Theorem; 

\[ (1.118) \]

4. \( \frac{\partial^p}{\partial r^p} g(r,\theta,\lambda) \bigg|_{r=R} \leftrightarrow (-1)^p \frac{(n+1)\cdots(n+p)}{R^p} G_{n,m} \)

radial differentiation of a harmonic function. 

\[ (1.119) \]

The proof of Parseval’s Theorem (1.118) comes directly from the orthogonality of the spherical harmonic functions, equation (1.106). Property 4 applies to the radial extension of a function on the sphere according to potential theory. That is, we assume that the function, \( g(\theta, \lambda) \), represents the boundary values on the sphere (radius, \( R \)) of a function, \( g(r,\theta,\lambda) \), that is harmonic in the space external to the sphere (recall equation (1.74)):

\[ \nabla^2 g(r,\theta,\lambda) = 0, \quad r > 0, \quad (1.120) \]

where, in this case, the Laplacian operator is given by

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \lambda^2}. \quad (1.121) \]

It can be shown that the solution to the partial differential equation (1.120) is given by

\[ g(r,\theta,\lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left( \frac{R}{r} \right)^{n+1} G_{n,m} \bar{V}_{n,m}(\theta,\lambda). \quad (1.122) \]

We see that \( \{G_{n,m}\} \) is the Legendre spectrum of the function, \( g_\theta(\theta, \lambda) = g(R,\theta,\lambda) \), on the sphere of radius, \( R \); and that \( \{(R/(R+h))^{n+1} G_{n,m}\} \) is the Legendre spectrum of the function, \( g_{R+h}(\theta, \lambda) = g(R+h,\theta,\lambda) \), on the sphere of radius, \( R+h, \ h \geq 0 \). Property 4, above, is easily verified by applying the derivatives to equation (1.122). It is noted that the Legendre spectrum is defined for derivatives of functions with respect to longitude, but not with respect to co-latitude. The convergence of the meridians on the sphere is the essential reason for this, as well, as other properties (such as for convolutions) that do not extend from the Cartesian to the spherical coordinate system. Similarly, many of our fundamental examples (Section 1.5) must be modified to accommodate this spherical geometry.
1.12 Some Important Examples

The “rectangular” function in the case of functions on the sphere is defined by necessity (for applications) in terms of just one variable, the angular distance from the center of spherical cap, as

\[
    b_s(\psi) = \begin{cases} 
        \frac{4\pi}{\sigma_s}, & \psi < \psi_s; \\
        0, & \psi > \psi_s; 
    \end{cases} 
\]  

(1.123)

where \( \sigma_s \) is the area of the spherical cap \( \{ \psi \mid 0 \leq \psi \leq \psi_s \} \) with radius \( \psi_s \) (see Figure 1.6):

\[
    \sigma_s = \int_0^{2\pi} \int_0^{\psi_s} d\sigma = 2\pi \int_0^{\psi_s} \sin \psi \, d\psi = 2\pi (1 - \cos \psi_s). 
\]  

(1.124)

Using a recursion formula for Legendre polynomials,

\[
    (2n+1) P_n(y) = \frac{d}{dy} \left( P_{n+1}(y) - P_{n-1}(y) \right), \quad n > 0, 
\]  

(1.125)

the spectrum of \( b_s \) is found to be, using equation (1.113),

\[
    B_n = \frac{4\pi}{2\sigma_s} \int_0^{\psi_s} P_n(\cos \psi) \sin \psi \, d\psi 
    = \frac{1}{1 - \cos \psi_s} \frac{1}{2n+1} \left( P_{n-1}(\cos \psi_s) - P_{n+1}(\cos \psi_s) \right), \quad n \geq 1 
\]  

(1.126)

with

\[
    \beta_0 = 1, \quad \beta_1 = \frac{1}{2} (1 + \cos \psi_s). 
\]  

(1.127)
In the geodetic literature, the spectral components, $B_n$, are denoted $\beta_n$ ($B_n \equiv \beta_n$) and are called “Pellinen smoothing factors;” we’ll see why they are “smoothing” factors later on. Recursion formulas exist for $\beta_n$; they are derived from corresponding recursion formulas for the Legendre polynomials, $P_n$. For example, substituting the recursion formula

$$ (n+1)P_{n+1}(y) = (2n+1)yP_n(y) - nP_{n-1}(y), \quad n \geq 1, $$

(1.128)

into the integral of equation (1.126), and using equation (1.125), we obtain

$$ \beta_n = \frac{2n-1}{n+1} \beta_{n-1} \cos \psi_s - \frac{n-2}{n+1} \beta_{n-2}, \quad n \geq 2, $$

(1.129)

with starting values given by (1.127). A plot of these smoothing factors for $\psi_s = 1^\circ$ is shown in Figure 1.7. Note that the first zero-crossing occurs at $n = 219 = 180^\circ/\psi_s$ (roughly). The similarity to the sinc function (Figure 1.2), which has its first zero-crossing at $1/T$, is evident.
Finally, we can define also the Dirac function, again, necessarily only as a “function” of \( \psi \):

\[
\begin{align*}
\text{a) } & \quad \delta_s(\psi) = 0, \quad \text{if} \quad \psi \neq 0; \\
\text{b) } & \quad \frac{1}{4\pi} \int \int_{\sigma} \delta_s(\psi) \, d\sigma = 1; \\
\text{c) } & \quad \frac{1}{4\pi} \int \int_{\sigma} \delta_s(\psi) \, g(\theta',\lambda') \, d\sigma = g(\theta,\lambda);
\end{align*}
\]

where from spherical trigonometry (see also Figure 1.6):

\[
\cos\psi = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\lambda - \lambda'),
\]

and \( \psi \) is the angle at the center of the unit sphere between \((\theta,\lambda)\) and \((\theta',\lambda')\). The Legendre spectrum of \( \delta_s(\psi) \) is given by
\[ D_n = \frac{1}{2} \int_0^{\pi} \delta_s(\psi) P_n(\cos\psi) \sin\psi \, d\psi \]
\[ = \frac{1}{4\pi} \int_0^{2\pi} \delta_s(\psi) P_n(\cos\psi) \, d\sigma \]
\[ = P_n(\cos 0) = 1, \quad \text{for all} \ n; \] 

where, in this case, \( \psi \), trivially is the angle “between 0 and \( \psi \”).

### 1.13 From Sphere to Plane

There is an approximate relationship between the 1-D Legendre transform and the Hankel transform for local applications (i.e., the sphere is approximated by a plane, locally). This allows the transition from the spherical spectral domain to the Cartesian spectral domain and from harmonic degree to spatial frequency on the plane. Using the asymptotic relationship between Legendre polynomials and Bessel functions, Forsberg (OSU Report No.356, 1984) derives

\[ \omega = 2\pi f \approx \frac{1}{R} \left( n + \frac{1}{2} \right), \]

where \( R \) is the mean radius of the Earth.

An alternative formula was given by Eckhardt (1983). In Cartesian coordinates, consider a function, \( v(x, y, z) = u(x, y) w(z) \), harmonic in the half space \( z \geq 0 \), i.e., \( \nabla^2 v = 0 \). It is readily shown that the part \( u(x, y) \) satisfies the differential equation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) + \omega^2 u(x, y) = 0, \]  

where \( \omega \) is some (real) constant with respect to \( u(x, y) \). Solutions of equation (1.136) are harmonic oscillators with radian frequency, \( \omega \). Analogously, in spherical coordinates, a harmonic function can be separated into \( v(r, \theta, \lambda) = p(r) q(\theta, \lambda) \), where \( q(\theta, \lambda) \) satisfies the differential equation

\[ \frac{1}{R^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \lambda^2} \right) q(\theta, \lambda) + \frac{n(n+1)}{R^2} q(\theta, \lambda) = 0. \]

The solutions of this equation are the spherical harmonic functions of degree, \( n \). The derivatives in equation (1.137) are equivalent to the derivatives in equation (1.136) in a local coordinate.
system with horizontal and vertical directions (say, north, east, down). Therefore, by comparing (1.136) and (1.137), we may approximate:

\[
\omega^2 \approx \frac{n(n+1)}{R^2} \tag{1.138}
\]

which is Eckhardt’s alternative relationship.

Note that Forsberg’s approximation does not work for \( n = 0 \), whereas, Eckhardt’s does. However, both approximations are found to be equally useful for the higher degrees (they are based, after all, on a local approximation); and both are generally better than the approximation \( \omega = n/R \), although for convenience, we often use the latter.

### 1.14 Examples of Transforms

In this section, several of the more common Fourier (1-D and 2-D), Hankel, and 1-D Legendre transforms are given. Most can be obtained by consulting a good Table of Integrals, e.g., Gradshteyn and Ryzhik (1980). Some of these were encountered already in the discussions of previous sections; and others are easily derived from these using the properties of transforms. Still others, especially the Legendre transform pairs, are familiar to geodesists. In each case, we use the generic symbol \( \leftrightarrow \) to denote the transform pair, and the type of transform should be obvious from the associated independent variables.

1. \( b_1(t) \leftrightarrow \text{sinc}(f) \) \hspace{1cm} (see equation (1.40)) \hspace{1cm} (1.139)

2. \( \delta(t) \leftrightarrow 1 \) \hspace{1cm} (see equation (1.49)) \hspace{1cm} (1.140)

3. \( e^{-\pi f^2} \leftrightarrow e^{-\pi f'^2} \) \hspace{1cm} (1-D) \hspace{1cm} (1.141)

4. \( e^{-|t|} \leftrightarrow \frac{2}{1+(2\pi f)^2} \) \hspace{1cm} (1-D) \hspace{1cm} (1.142)

5. \( b_1(t_1, t_2) \leftrightarrow \text{sinc}(f_1)\text{sinc}(f_2) \) \hspace{1cm} (see equation (1.69)) \hspace{1cm} (1.143)

6. \( \delta(t_1, t_2) \leftrightarrow 1 \) \hspace{1cm} (see equation (1.71)) \hspace{1cm} (1.144)

7. \( \frac{1}{(x_1^2 + x_2^2 + a^2)^{3/2}} \leftrightarrow \frac{2\pi}{a} e^{-2\pi a(f_1^2 + f_2^2)^{1/2}} ; \quad a > 0 \) \hspace{1cm} (1.145)
8. \[
\frac{x_1}{(x_1^2 + x_2^2)^{3/2}} \leftrightarrow -\frac{i2\pi f_1}{(f_1^2 + f_2^2)^{1/2}}; \quad \frac{x_2}{(x_1^2 + x_2^2)^{3/2}} \leftrightarrow -\frac{i2\pi f_2}{(f_1^2 + f_2^2)^{1/2}}
\] (1.146)

9. \[b_*(r) \leftrightarrow \frac{1}{\pi} J_1(2\pi f a) \quad \text{(see equation (1.100))} \] (1.147)

10. \[\frac{1}{r} \leftrightarrow \frac{1}{f} \quad \text{(Hankel transform)} \] (1.148)

11. \[\frac{1}{(r^2 + a^2)^{1/2}} \leftrightarrow \frac{1}{f} e^{-2\pi af} \quad \text{(Hankel transform)} \] (1.149)

12. \[e^{-\pi r^2} \leftrightarrow e^{-\pi f^2} \quad \text{(Hankel transform)} \] (1.150)

13. \[e^{-ar} \leftrightarrow \frac{2\pi a}{(4\pi^2 f^2 + a^2)^{3/2}} \quad \text{(Hankel transform)} \] (1.151)

14. \[\frac{1}{r} e^{-ar} \leftrightarrow \frac{2\pi a}{(4\pi^2 f^2 + a^2)^{1/2}} \quad \text{(Hankel transform)} \] (1.152)

15. \[r^2 e^{-ar} \leftrightarrow \frac{1}{\pi} \left(\frac{1}{\pi} - f^2\right) e^{-\pi f^2} \quad \text{(Hankel transform)} \] (1.153)

16. \[\frac{1}{\ell} = \frac{1}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\psi}} \leftrightarrow \frac{1}{2n+1} \frac{1}{R_1} \left(\frac{R_1}{R_2}\right)^{n+1}, \quad R_2 > R_1, \quad \text{(1-D)} \] (1.154)

17. \[\frac{1}{\ell} \leftrightarrow \frac{1}{2n+1} \frac{1}{R_1} \left(\frac{R_1}{R_2}\right)^{n+1} Y_{n,m}(\theta',\lambda'), \quad R_2 > R_1 \quad \text{(2-D)} \] (1.155)

18. \[\frac{R_1(R_2^2 - R_1^2)}{\ell^3} \leftrightarrow \left(\frac{R_1}{R_2}\right)^{n+1}, \quad R_2 > R_1 \quad \text{(1-D)} \] (1.156)

19. \[\frac{2R_1}{\ell} + \frac{R_1}{R_2} - \frac{3R_1}{R_2^2} \cos\psi \left(\frac{\ell + R_2 - R_1\cos\psi}{2R_2}\right) \leftrightarrow \begin{cases} \frac{1}{n-1} \left(\frac{R_1}{R_2}\right)^{n+1}, & n \geq 2; \\ 0, & n = 0,1; \end{cases} \quad R_2 > R_1 \] (1.157)

This is the 1-D Legendre transform of the generalized Stokes’ function \(S(R_2, \psi)\).